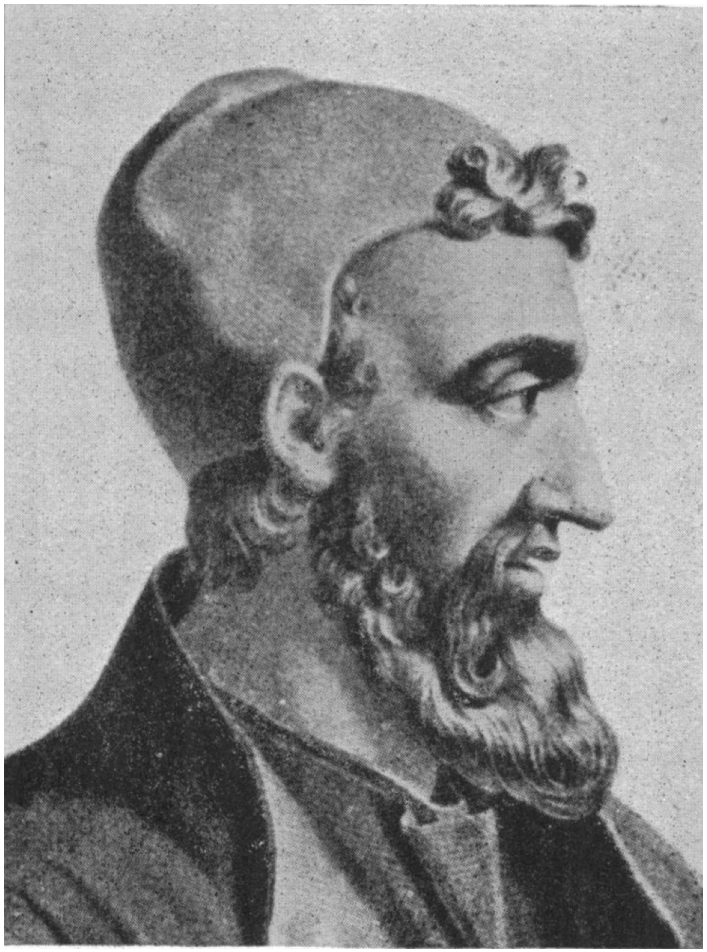
Arterial wave propagation

- Notions of pressure and flow wave propagation
- 1-D flow models
- Pulse wave velocity
- Input impedance and characteristic impedance
- Wave separation into forward and backward running waves
- Transfer function
- Global models (3- and 4-element Windkessel models)

Feeling the pulse...

A consultation by Susruta, the Hindu physician, who is portrayed feeling the pulse

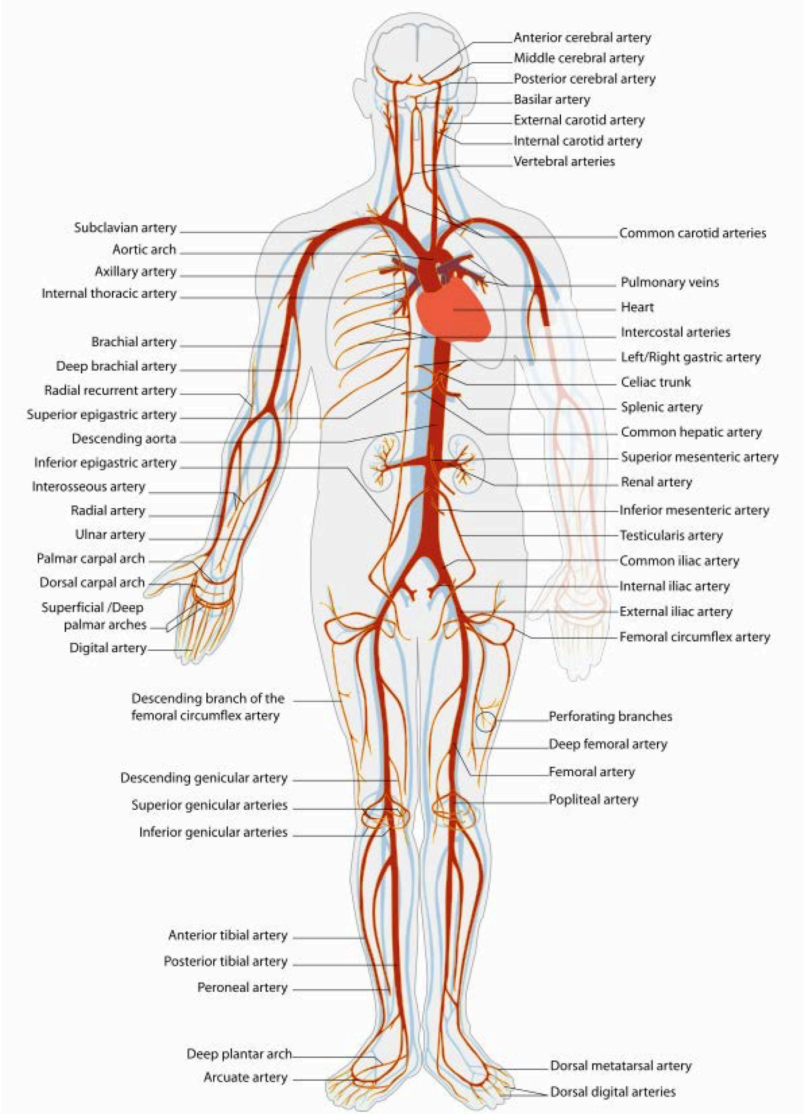


Claudius Galenus (A.D.131-200). The greatest sphygmologist of antiquity and author of 18 books on the arterial pulse

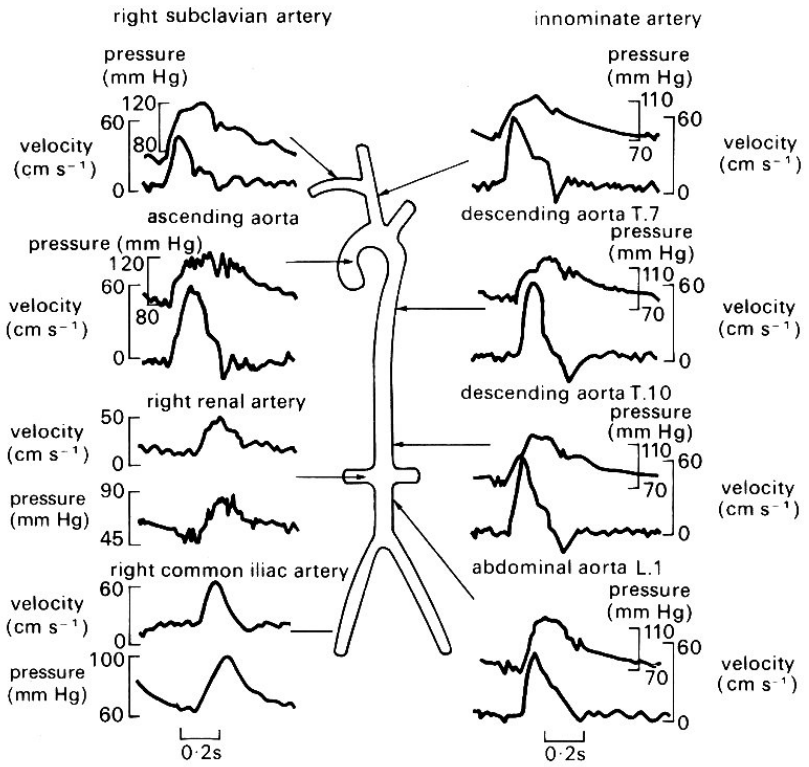
Wave propagation in the arterial system

Arterial system:

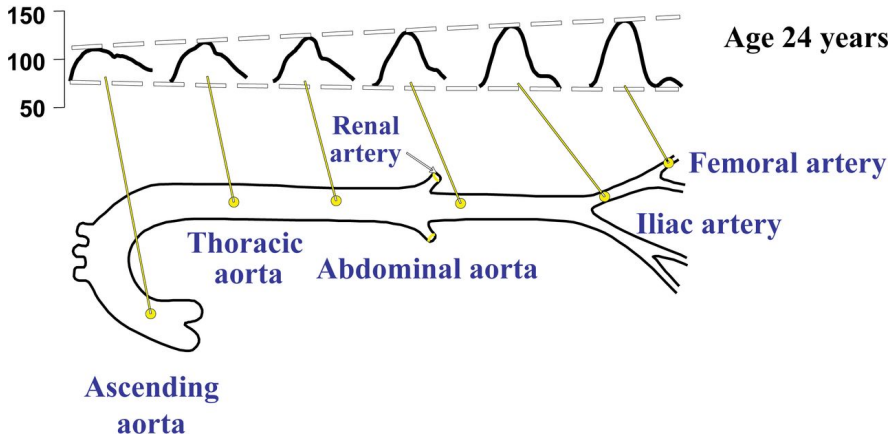
- Complex network of viscoelastic tubes
- Complicated geometry
- Nonlinear elastic properties
- Non-newtonian fluids, large variation of Reynolds numbers



Wave propagation

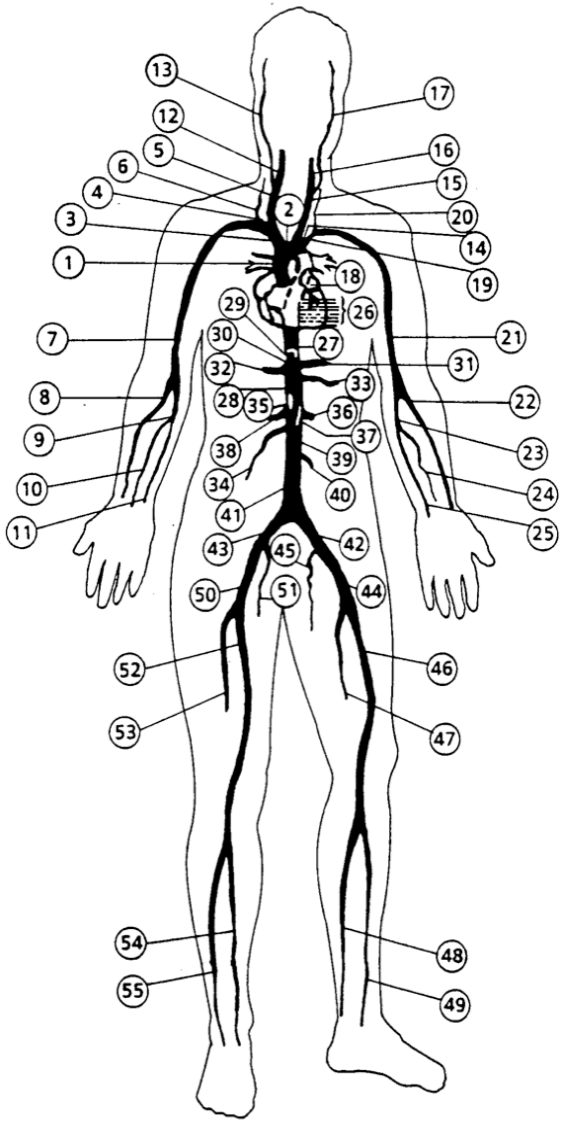


Pressure and flow waves are distinctly different in different locations of the arterial tree



- Pressure waves are “amplified” as we move along the aorta.
- Systolic pressure is, in general, higher in peripheral arteries as compared to the aorta

1-D model of the arterial system



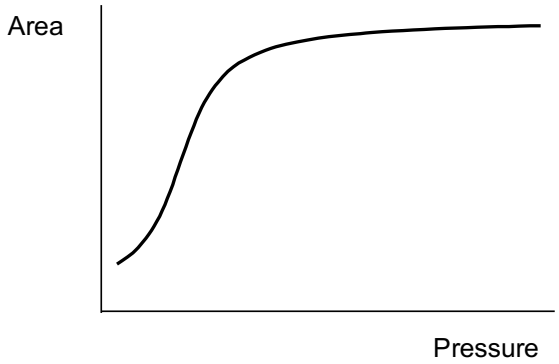
Continuity

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0$$

Momentum

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\lambda\frac{Q^2}{A}\right)=-\frac{A}{\rho}\frac{\partial P}{\partial x}+\frac{\pi D\tau_w}{\rho}$$

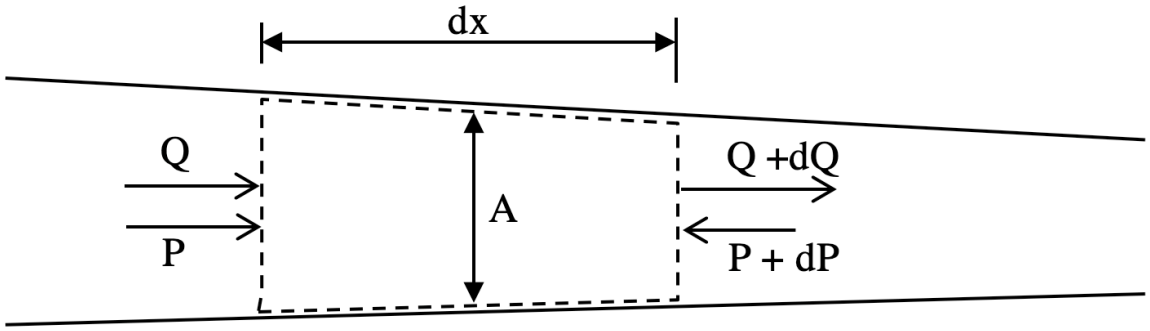
Constitutive law



- Three equations (Continuity, momentum and constitutive law A=f(p))
- Three unknowns:
  1. Area (A),
  2. Pressure (P) and
  3. Flow (Q)

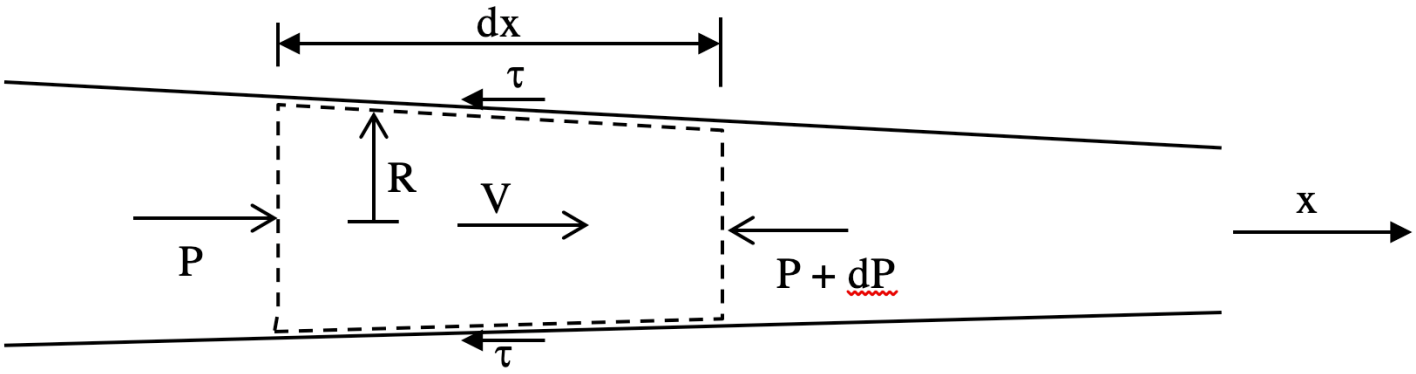
1-D model of the arterial system

Conservation of mass



$$Q_{in} - Q_{out} = \frac{dV}{dt} \Rightarrow Q - (Q + dQ) = \frac{d(dx \cdot A)}{dt} = dx\frac{\partial A}{\partial t} \quad \Rightarrow \quad \frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \qquad (1)$$

Conservation of momentum



Assumption: no radial velocity component (v << u)

Newton's 2nd law:

$$\sum F_x = m \cdot a_x$$

$$\Rightarrow \left[P - (P + dP) \right] \cdot A - \tau \cdot (2\pi R \cdot dx) = (\rho \cdot A \cdot dx) \frac{\partial V}{\partial t}$$

Note: $Q = A\,V$

$$dQ = d(A \cdot V) = dA \cdot V + A \cdot dV \approx A \cdot dV$$

$$\Rightarrow -\frac{\partial P}{\partial x} \cdot A - \tau \cdot 2\pi R = \rho \cdot \frac{\partial Q}{\partial t}$$

$$\Rightarrow \quad \frac{\partial Q}{\partial t} = -\frac{A}{\rho}\frac{\partial P}{\partial x} - \tau \cdot \frac{2\pi R}{\rho} \qquad (2a)$$

We have neglected convective acceleration (tapering) effects. If included, Eq. (2a) becomes:

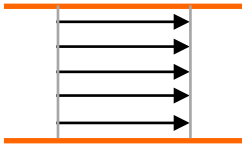
$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\lambda\frac{Q^2}{A}\right)=-\frac{A}{\rho}\frac{\partial P}{\partial x}-\tau\cdot\frac{2\pi R}{\rho} \qquad (2b)$$

Problem:

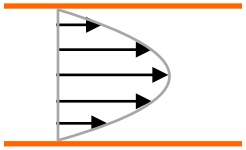
$$\left.\begin{aligned}\lambda&=\frac{1}{A}\int\frac{u^2}{V^2}dA\\ \tau&=\mu\left.\frac{\partial u}{\partial r}\right|_{r=R}\end{aligned}\right\}$$ both depend on the velocity profile shape factor.

λ {

= 1 for a flat velocity profile



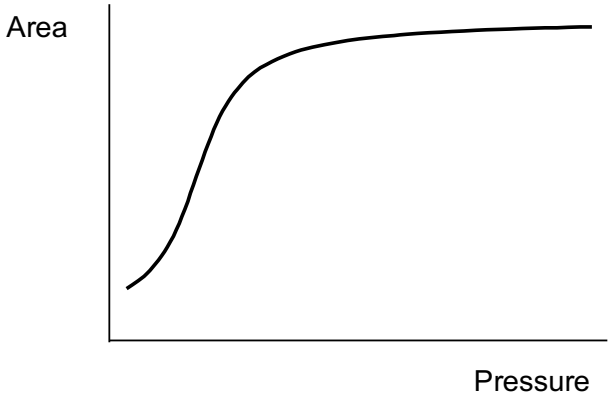
= 4/3 for a parabolic velocity profile



The wall shear stress is usually approximated by Poisseuille's law: $\tau=\dfrac{4\mu Q}{\pi R^3}$

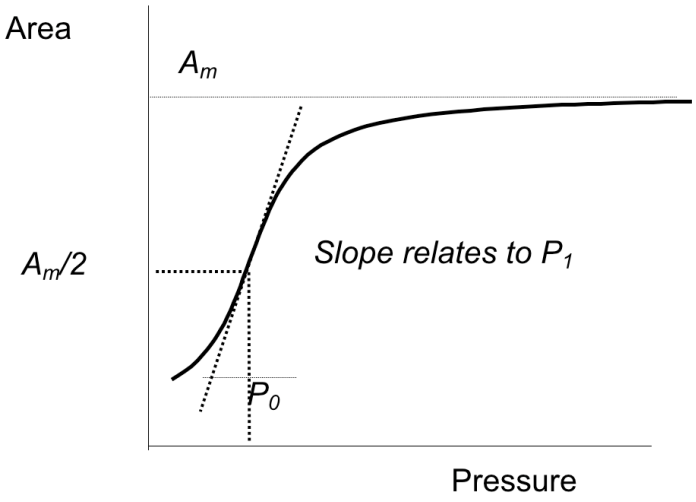
Constitutive equation

A constitutive equation gives the relationship between pressure and cross-sectional area: A = f(p).



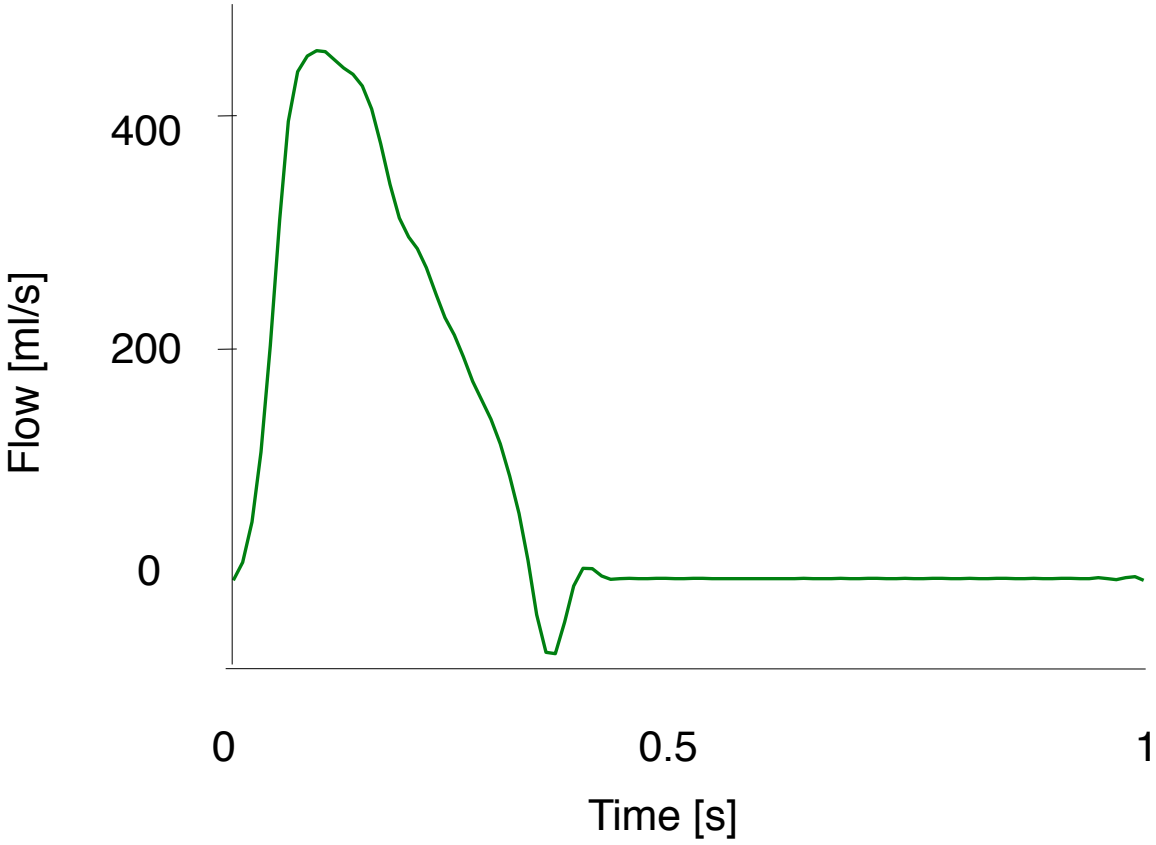
In analytical form, we may used the Langewouters formula:

$$A = A_m\left[\frac{1}{2} + tan^{-1}\left(\frac{P - P_o}{P_1}\right)\right]$$



Upstream boundary condition

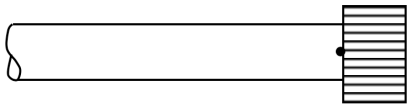
- Specify either the (aortic) pressure or flow waveform.
- Most often the aortic flow waveform (figurer below) is used as the upstream boundary condition



Downstream boundary conditions

- Specify either:
  1. pressure waveform
  2. flow waveform or
  3. the terminal (distal) impedance (Z_T)

- The terminal impedance model of vasculature beyond the terminal point and characterizes the relation between pressure and flow at the terminal point.

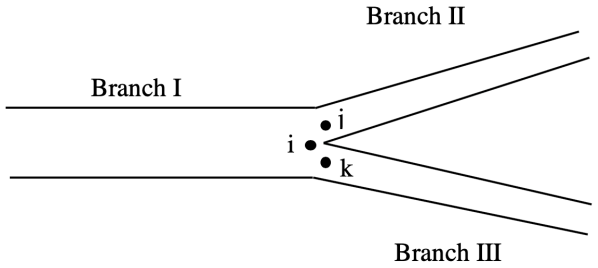


$$Z_T = \frac{P}{Q}$$

Boundary condition at bifurcation

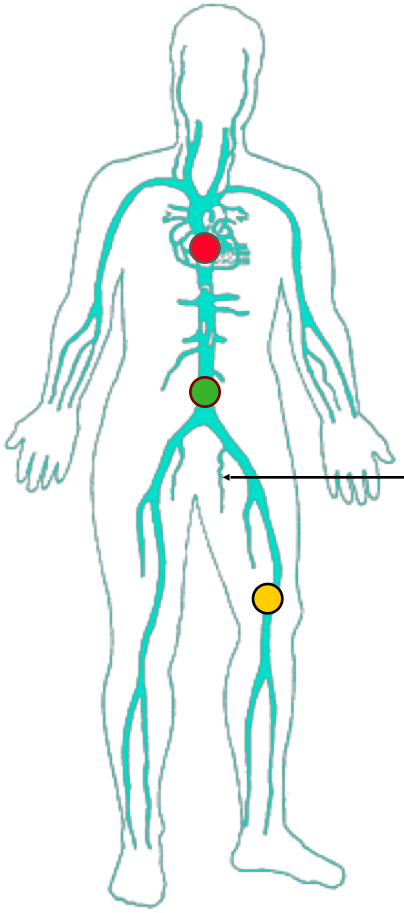
Apply a simple continuity in pressure and flow, such as described below.

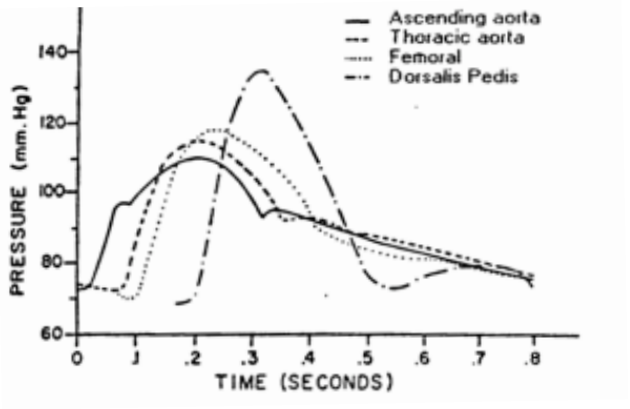
Indices:
i = last node of mother branch I
j = first node of daughter branch II
k = first node of daughter branch III



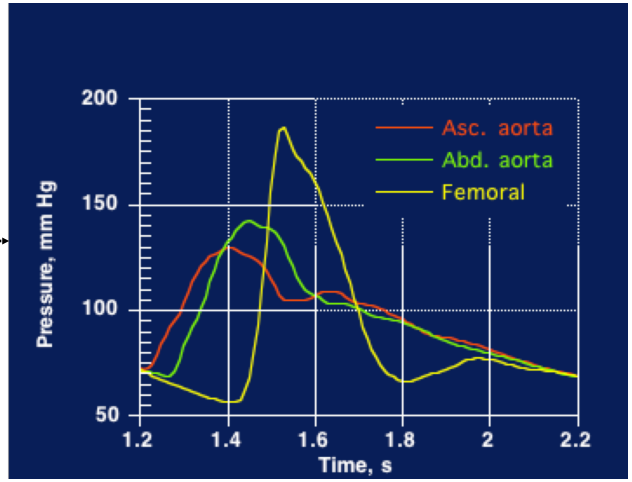
$$p_i = p_j = p_k$$
$$Q_i = Q_j + Q_k$$

1-D model predictions



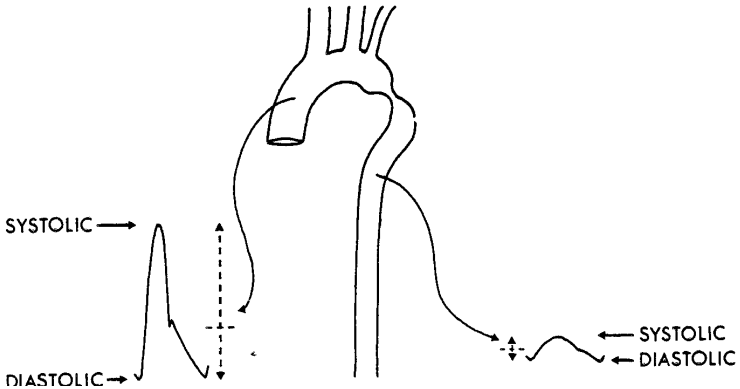


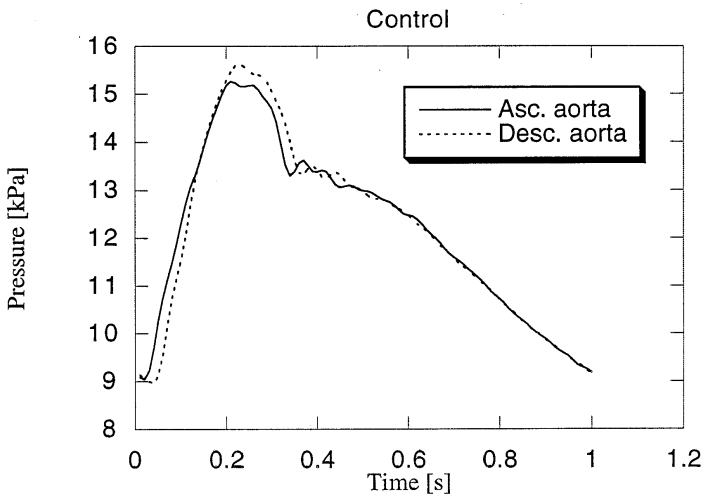
Human data (From Guyton, Textbook of Medical Physiology)



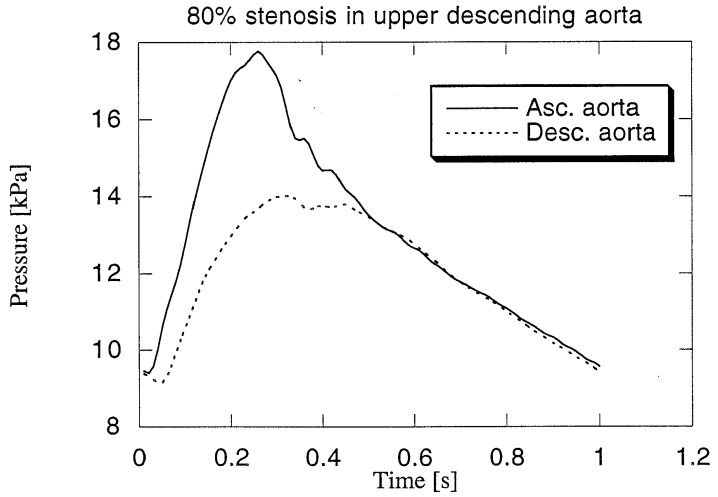
From computer simulation using an 1-D model of the human arterial system

Example: aortic stenosis





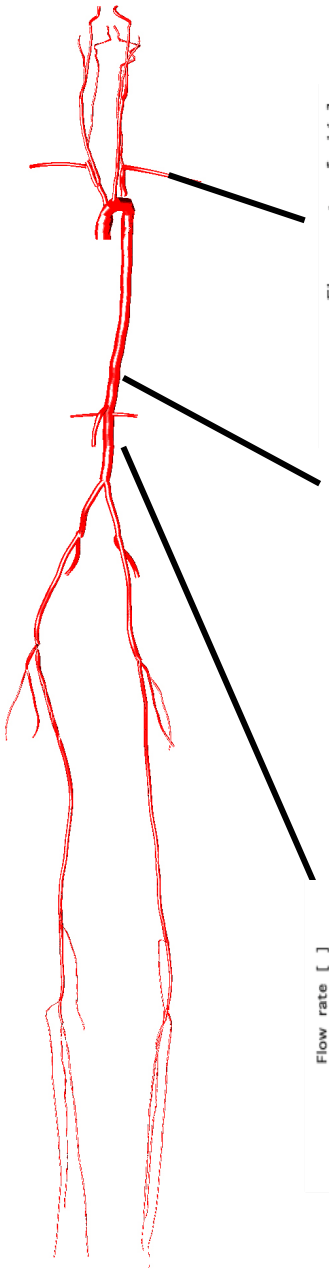
Distal to aortic stenosis:
- severe damping of wave
- severe damping of high frequencies
- systolic values most affected

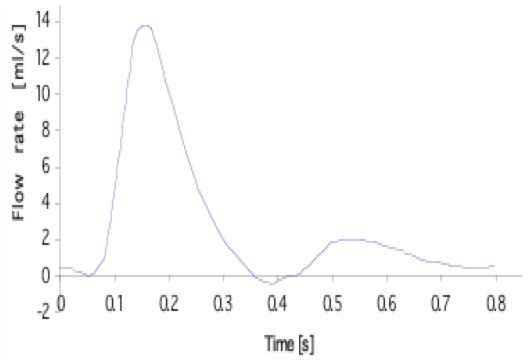


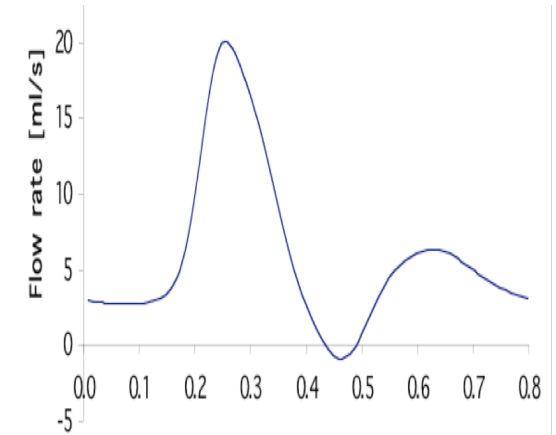
Flowrate waveforms
Main systemic arteries

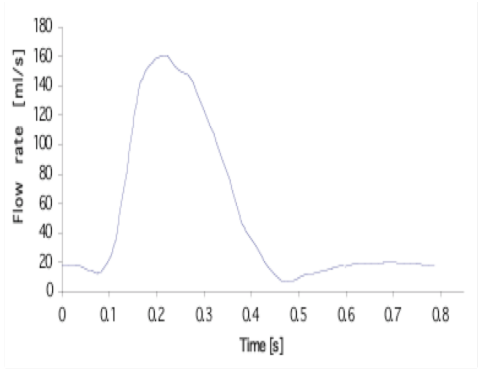
MRI

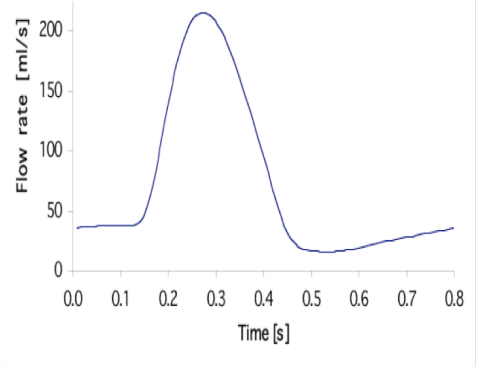
Model

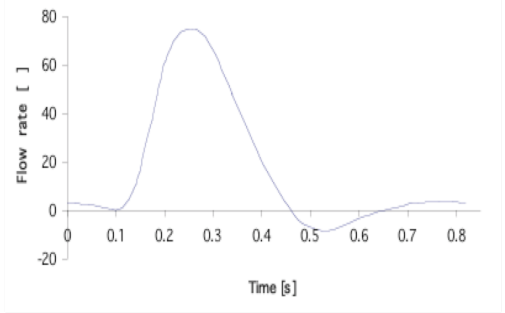


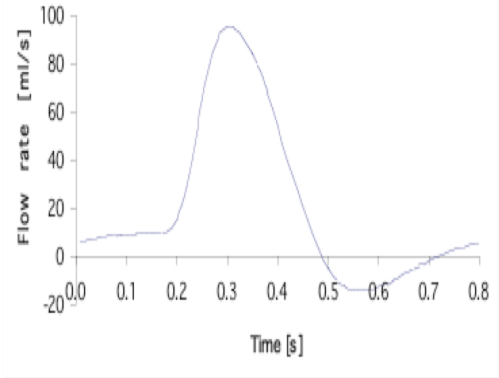




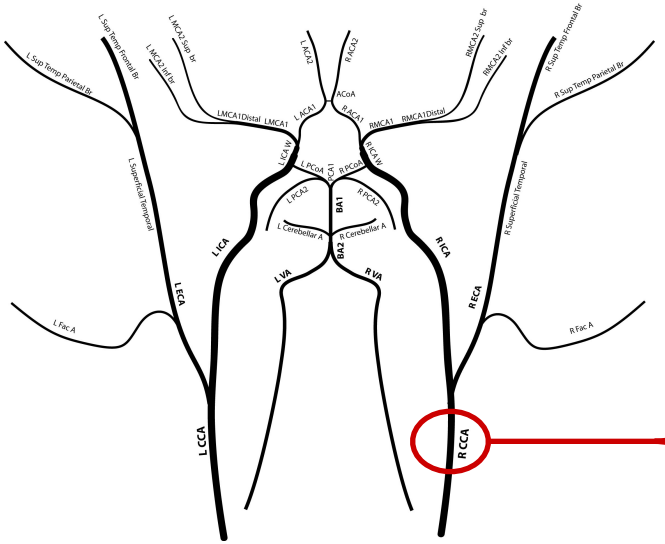




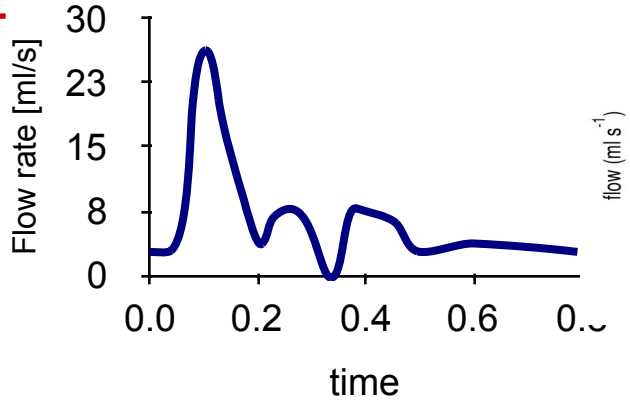




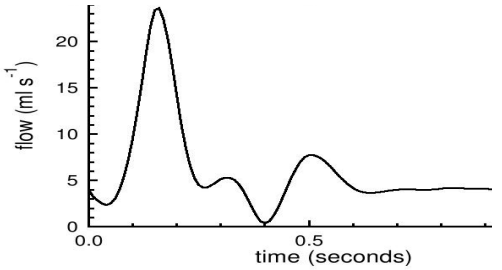
Model vs. measurements

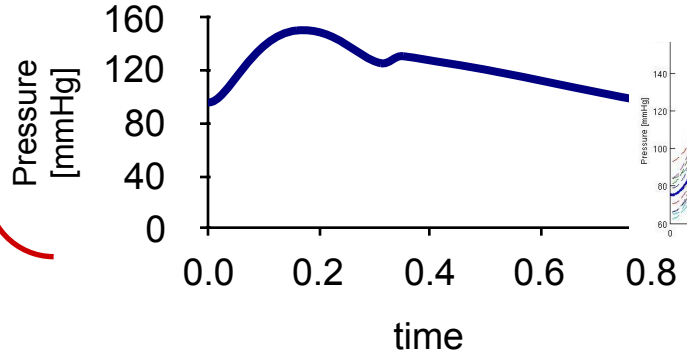


model

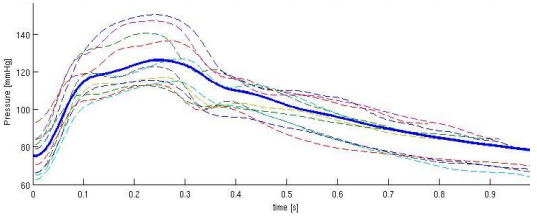


Flow rate (Doppler)





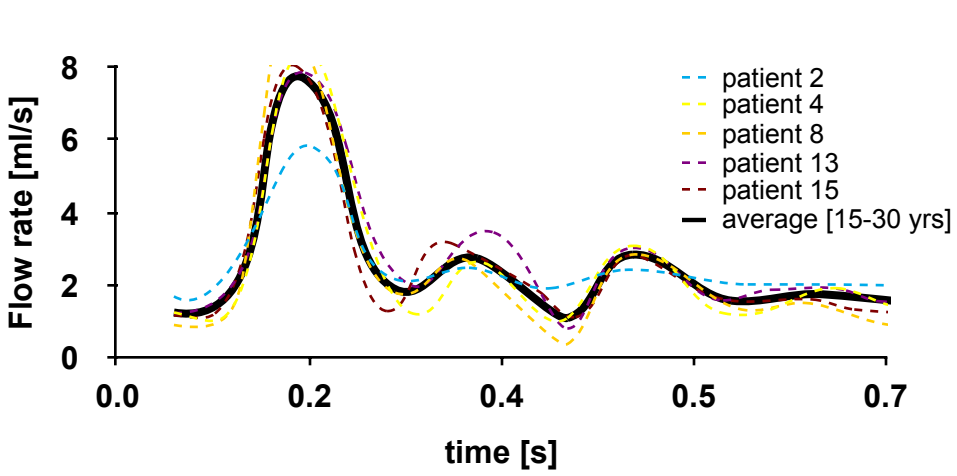
Pressure (tonometry)

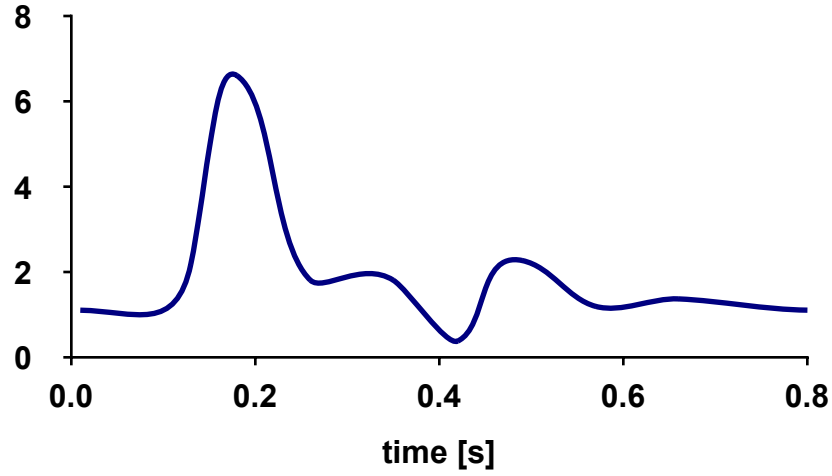


Measurements

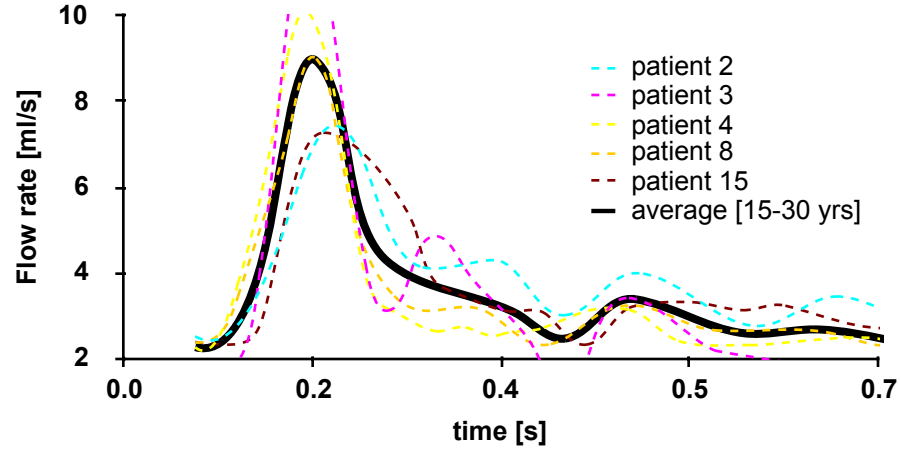
Model

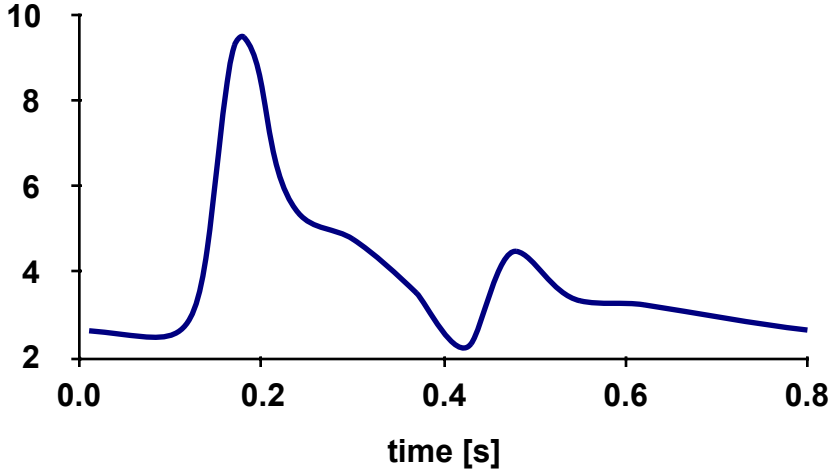
Left External Carotid Artery





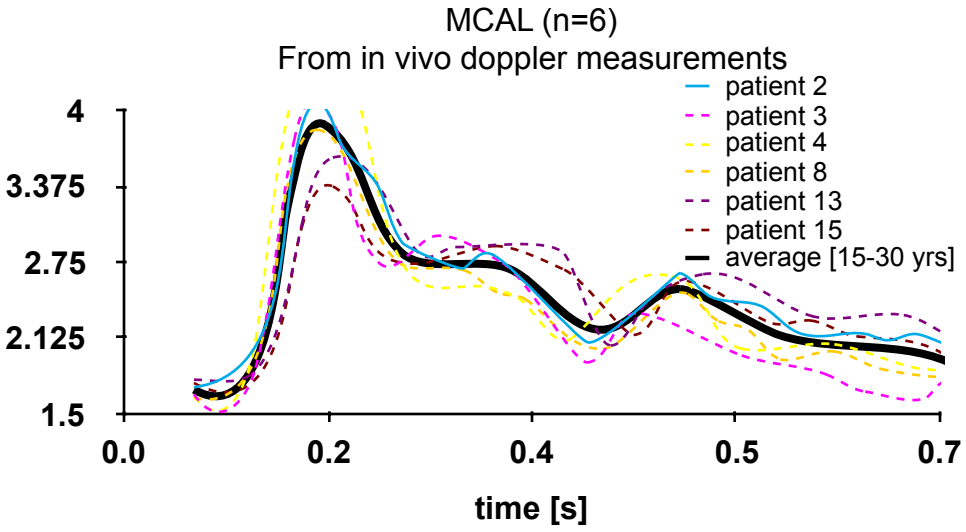
Left Internal Carotid Artery

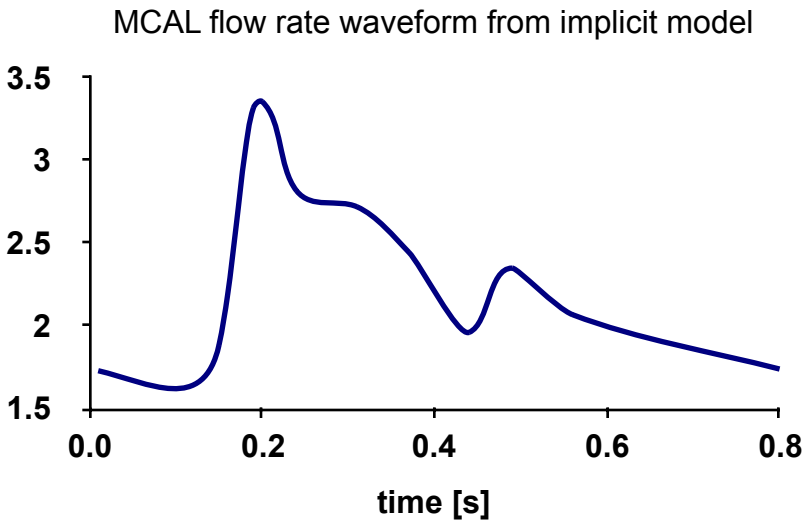




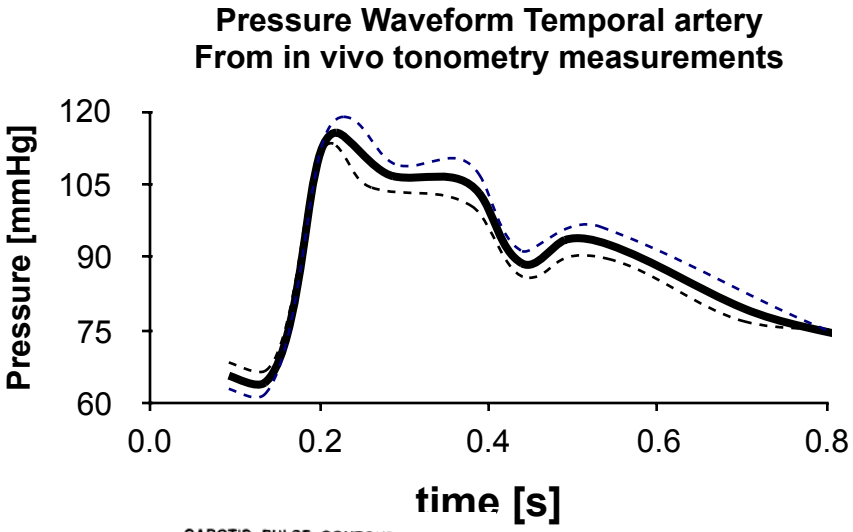
Flowrate waveforms

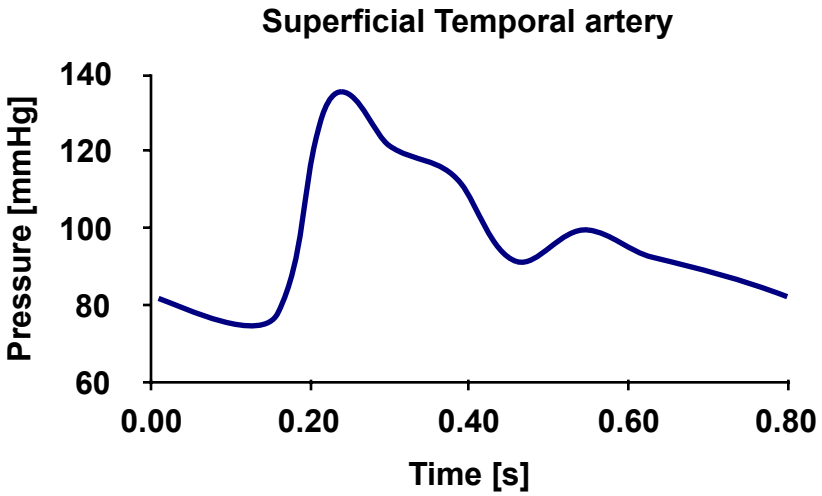
Middle cerebral Artery



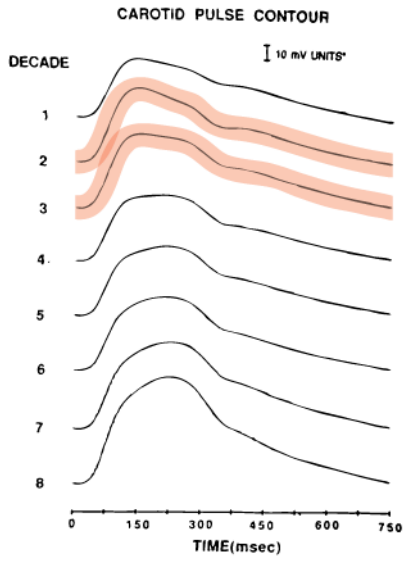


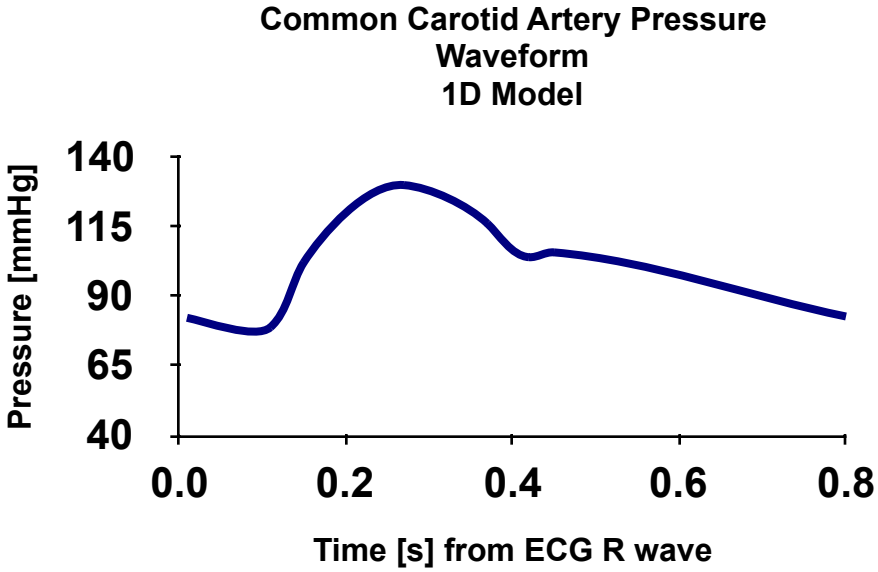
1D Model result



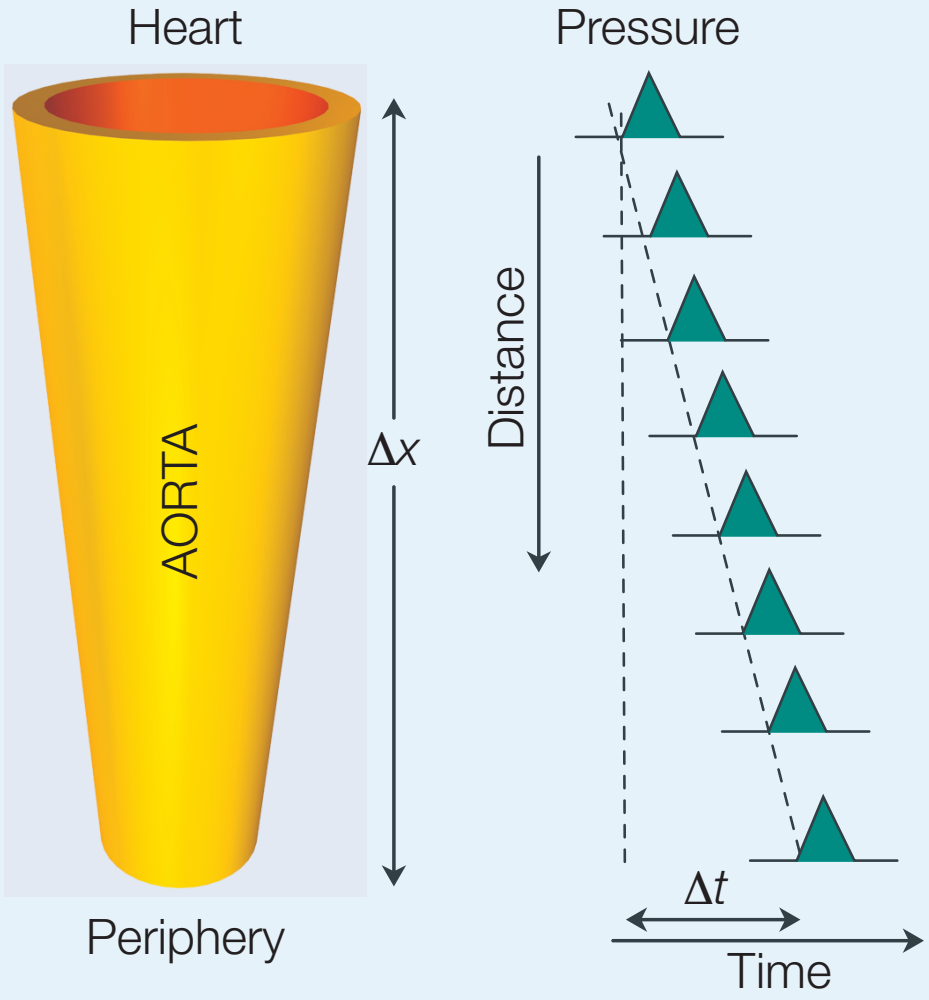


Carotid Artery





Wave travel & wave speed



Wave speed, c, is

$c = \Delta x/\Delta t$

Wave speed as as a function of area compliance, $C_A = \Delta A/\Delta P$, wall elastic modulus, E_{inc}, wall thickness, h, and blood density is:

$c = \sqrt{A/\rho C_A}$
(Newton-Young, or Frank, or Bramwell-Hill equation)

$c = \sqrt{E_{inc} h/\rho D}$

(Moens-Korteweg equation)

Relation of wave speed to arterial compliance

Continuity: $\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$

Momentum: $\frac{\partial Q}{\partial t} + \frac{A}{\rho}\frac{\partial P}{\partial x} + \tau\frac{2\pi R}{\rho} = 0$

If we neglect friction: $\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$

$\frac{\partial Q}{\partial t} + \frac{A}{\rho}\frac{\partial P}{\partial x} = 0$

The area compliance is defined as: $C_A = \frac{\partial A}{\partial P}$

$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial P}\frac{\partial P}{\partial t} = C_A\frac{\partial P}{\partial t}$

The 1D flow equations can be rewritten as:

$$\left.\begin{aligned} C_A\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\ \frac{\partial Q}{\partial t} + \frac{A}{\rho}\frac{\partial P}{\partial x} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} C_A\frac{\partial^2 P}{\partial t^2} + \frac{\partial^2 Q}{\partial x \partial t} &= 0 \\ \frac{\partial^2 Q}{\partial x \partial t} + \frac{A}{\rho}\frac{\partial^2 P}{\partial x^2} &= 0 \end{aligned}$$

$$\Rightarrow \frac{\partial^2 P}{\partial t^2} = \frac{A}{\rho C_A}\frac{\partial^2 P}{\partial x^2}$$ Wave equation!

where the wave speed is: $c^2 = \frac{A}{\rho C_A}$

or $c = \sqrt{\frac{A}{\rho}\frac{1}{C_A}}$

Newton-Young
or
Frank
or
Bramwell-Hill
equation

The wave speed, c, is often also called Pulse Wave Velocity, PWV

Relation of wave speed to arterial wall elasticity

Bramwell-Hill relation: $c = \sqrt{\frac{A}{\rho}\frac{1}{C_A}}$ (1)

Area compliance: $C_A = \frac{\partial A}{\partial p} = \frac{\partial(\pi R^2)}{\partial p} = 2\pi R \frac{\partial R}{\partial p}$ (2)

(1) ∧ (2) ⇒ $c = \sqrt{\frac{\pi R^2}{\rho}\cdot\frac{1}{2\pi R}\frac{\partial p}{\partial R}} = \sqrt{\frac{R}{2\rho}\frac{\partial p}{\partial R}}$ (3)

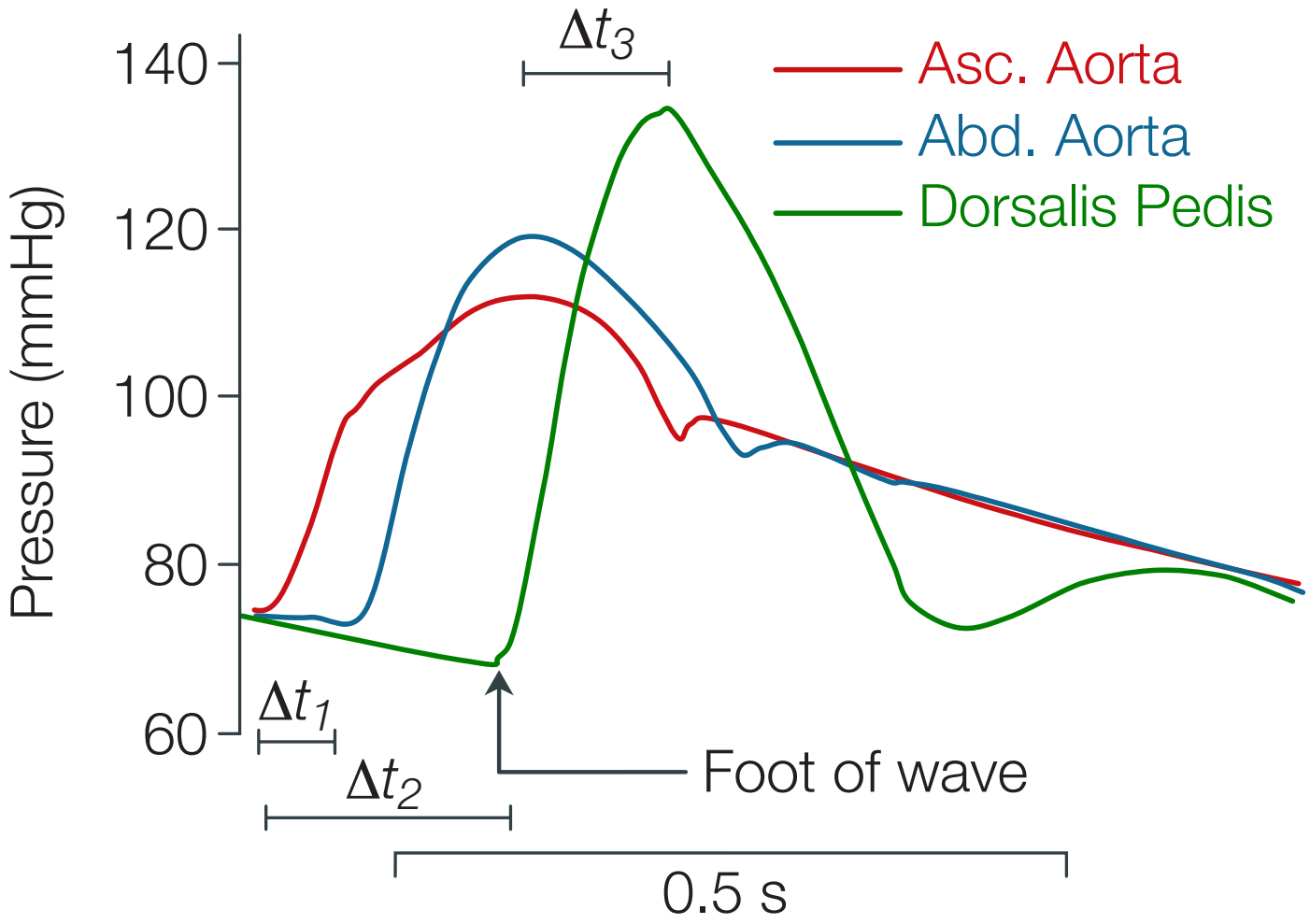
The incremental elastic modulus of the arterial wall is given by:

$E_{inc} = \frac{R^2}{h}\frac{\partial p}{\partial R} \Rightarrow \frac{\partial R}{\partial p} = \frac{R^2}{h}\cdot\frac{1}{E_{inc}}$ (4)

(3) ∧ (4) ⇒ $c = \sqrt{\frac{R}{2\rho}\frac{h\,E_{inc}}{R^2}} = \sqrt{\frac{E_{inc}\cdot h}{\rho\cdot 2R}}$

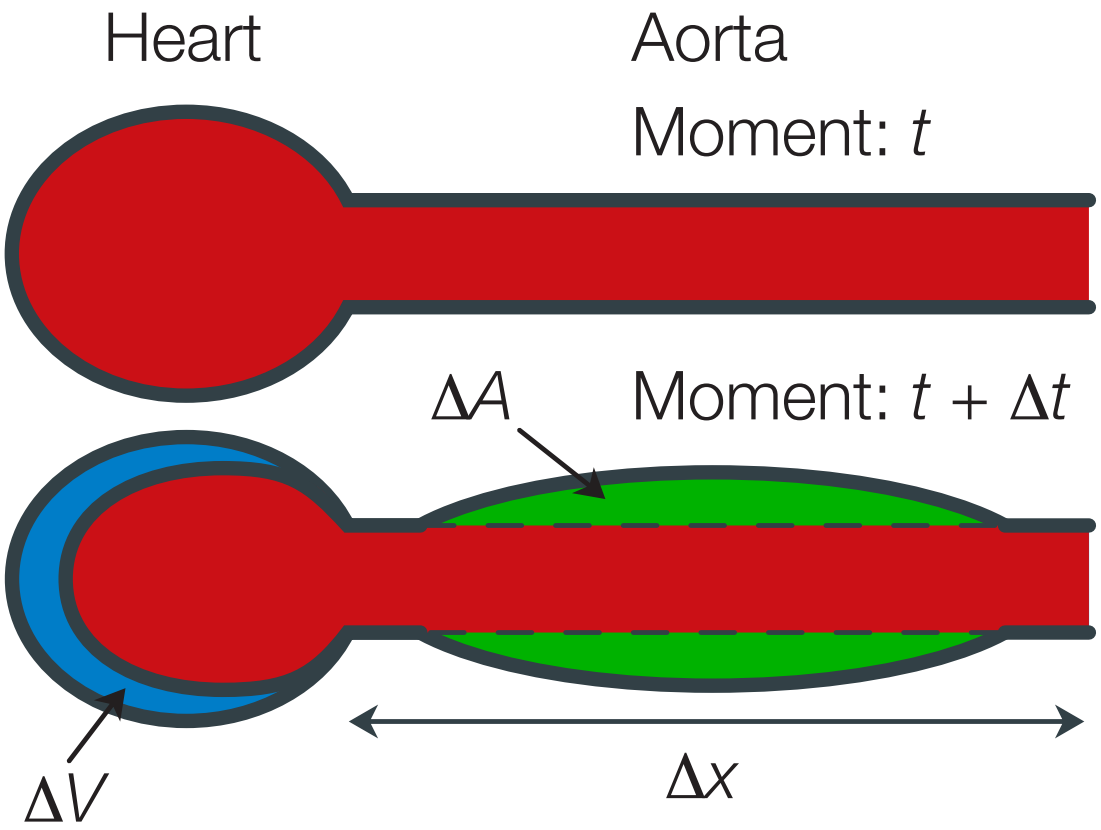
or $c = \sqrt{\frac{E_{inc}\cdot h}{\rho D}}$ Moens-Korteweg relation

Foot-to-foot method to derive wave speed



$$c = \frac{\Delta x}{\Delta t}$$ is valid only at the foot of the wave, were the wave is unidirectional (no reflected waves)

Derive PWV from the relation of area to flow



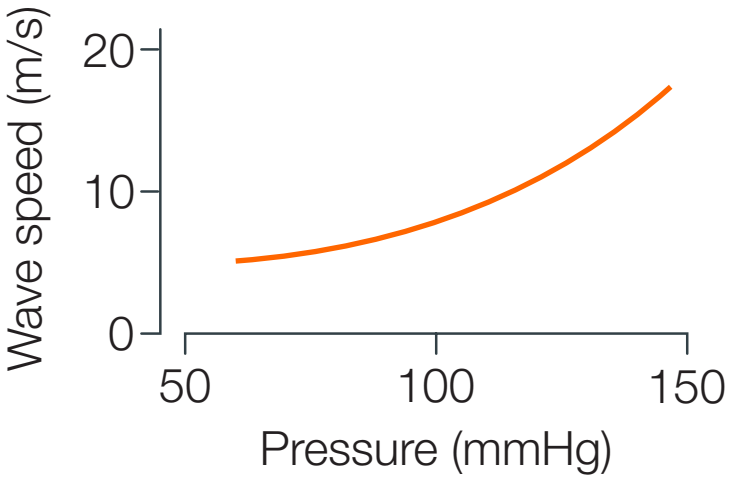
$$\Delta V = \Delta A \cdot \Delta x$$

$$\frac{\Delta V}{\Delta t} = \Delta A \cdot \frac{\Delta x}{\Delta t}$$

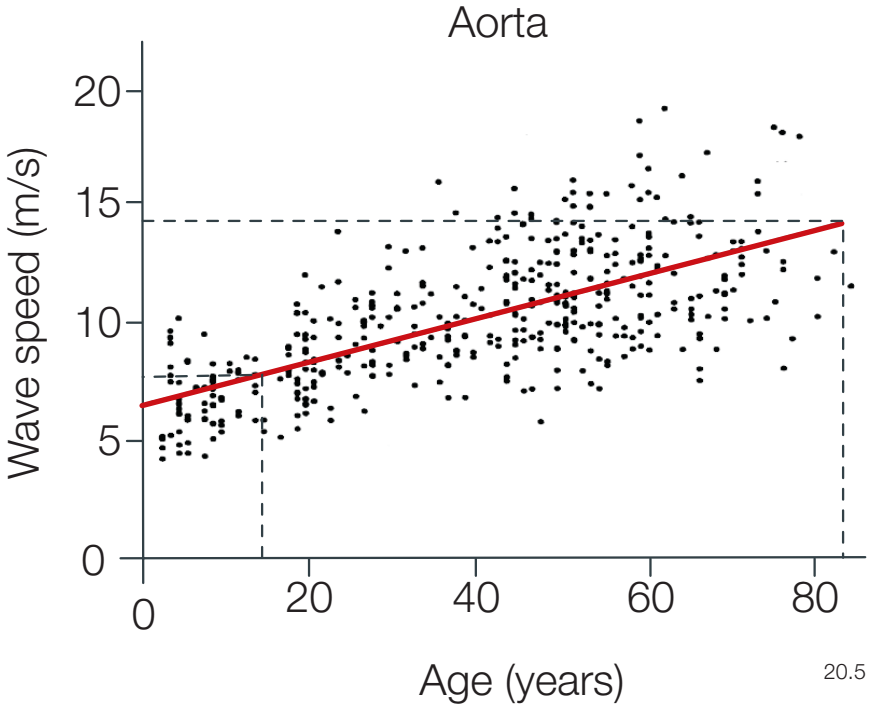
$$\Delta Q = \Delta A \cdot c$$

$$c = \frac{\Delta Q}{\Delta A}$$

PWV vs. Pressure and age

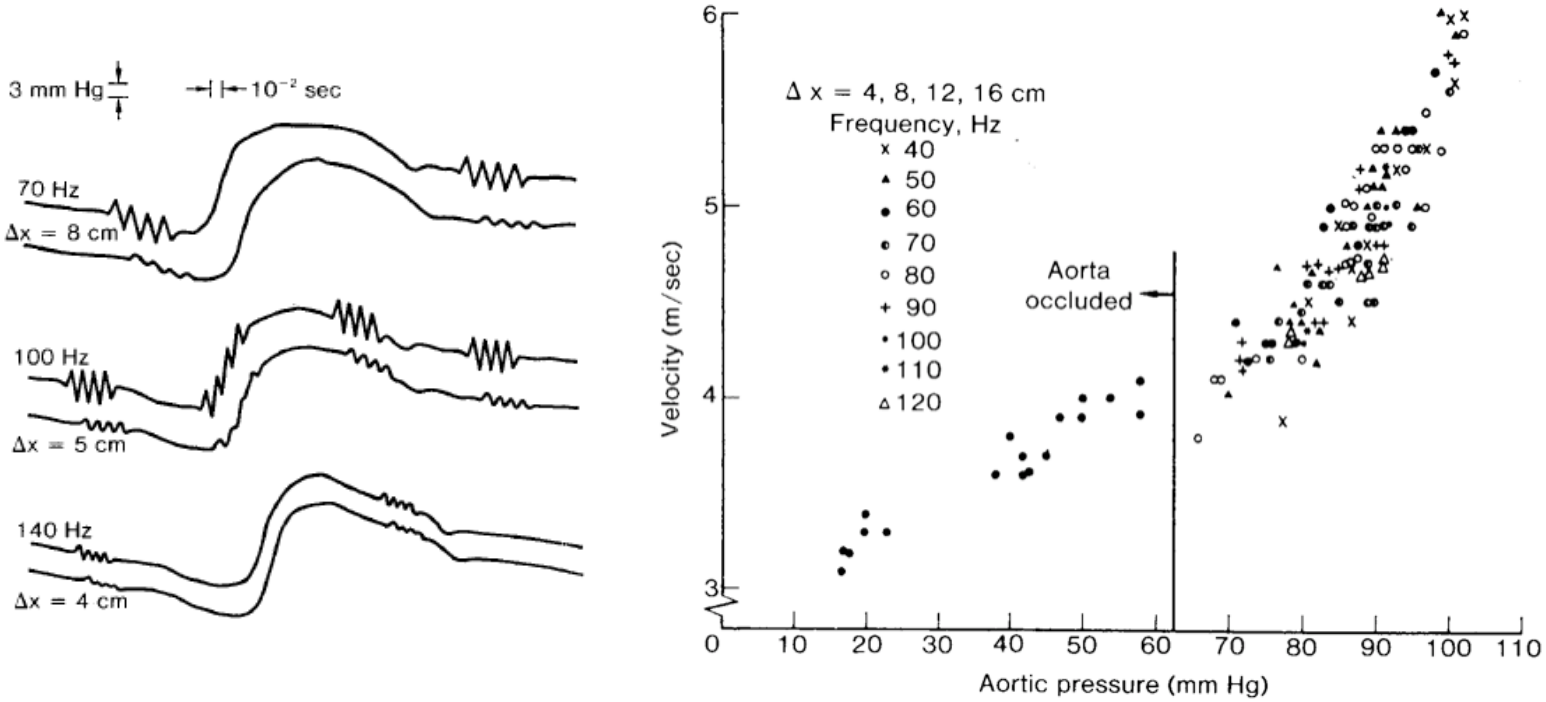


WAVE SPEED as a function of pressure derived from diameter and pressure measured noninvasively in the human brachial artery.



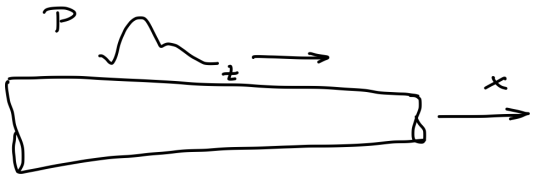
PULSE WAVE VELOCITY of the aorta as a function of age in individuals with low prevalence of atherosclerosis in Beijing.

Pulse wave velocity (PWV)



Velocity of small transient waves imposed on the aorta of the dog
From Anliker et al., Circ. Res. 23: 539-551.

Wave propagation without attenuation



$$p(x,t) = p_0 \, e^{i(\omega t - kx)}$$

If λ = wave length, at $x = \lambda$ the phase is 2π

$$k\lambda = 2\pi \Rightarrow k = \frac{2\pi}{\lambda}$$

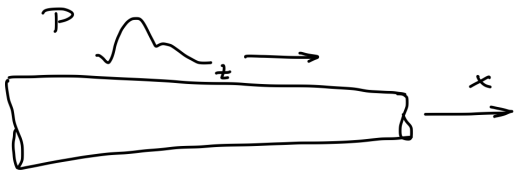
Also for a simple harmonic: $c = \lambda \cdot f$

$$\Rightarrow \quad k = \frac{2\pi f}{c} \quad \text{or} \quad k = \frac{\omega}{c}$$

Travelling wave without attenuation:

$$p(x,t) = p_0 \, e^{i\omega\left(t - \frac{x}{c}\right)}$$

Wave propagation with attenuation



$$P(x,t) = P_0 \, e^{-\alpha x} \, e^{i(\omega t - bx)}$$

attenuation coefficient

$P_0 \, e^{-\alpha x}$: Decreasing amplitude due to attenuation

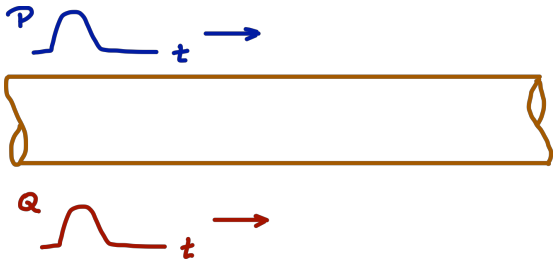
$$\Rightarrow P(x,t) = P_0 \, e^{i\omega t - (\alpha + ib)x} = P_0 \, e^{i\omega \left(t - \frac{\alpha + ib}{i\omega} x\right)} = P_0 \cdot e^{i\omega \left(t - \frac{x}{c}\right)}$$

where $c = \dfrac{i\omega}{\alpha + ib}$ is the complex wave velocity

We define also $\gamma = \alpha + ib = \dfrac{i\omega}{c}$ as the complex propagation coefficient

Womersley (1955, 1957) developed a complete theory for wave propagation of a viscous fluid in an elastic and viscoelastic tube, where γ is given as function of the Womersley number, arterial dimensions and wall viscoelastic properties.

Characteristic impedance, Z_c



$$Z_c = \frac{P}{Q}$$

Definition: Characteristic impedance, Z_c, is the ratio of pulsatile pressure to flow in case of unidirectional waves (no reflections)

Relation of Zc to geometry and elastic properties

Let $P(t)$ be a single harmonic wave $P(t) = P_0 e^{i\omega(t - \frac{x}{c})}$

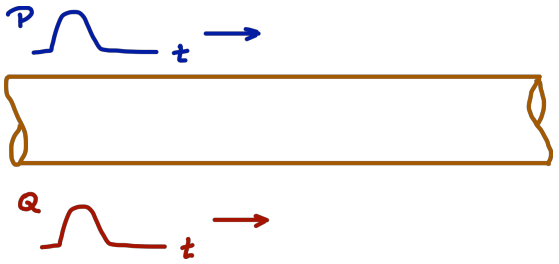
Continuity: $\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \Rightarrow \underbrace{\frac{\partial A}{\partial P}}_{C_A}\frac{\partial P}{\partial t} + \underbrace{\frac{\partial Q}{\partial P}}_{\frac{1}{Z_c}}\frac{\partial P}{\partial x} = 0 \Rightarrow C_A \frac{\partial P}{\partial t} + \frac{1}{Z_c}\frac{\partial P}{\partial x} = 0$

$$\Rightarrow C_A \cdot i\omega P_0 e^{i\omega(t - \frac{x}{c})} + \frac{1}{Z_c}\cdot\left(-\frac{i\omega}{c}\right)\cdot P_0 e^{i\omega(t-\frac{x}{c})} = 0 \Rightarrow C_A - \frac{1}{Z_c c} = 0$$

$$\Rightarrow Z_c = \frac{1}{c C_A} = \frac{1}{\sqrt{\frac{A}{\rho}\frac{1}{C_A}}\cdot C_A} = \sqrt{\frac{\rho}{A}\cdot\frac{1}{C_A}}$$

Since $c = \sqrt{\frac{A}{\rho}\frac{1}{C_A}} \Rightarrow \frac{Z_c}{c} = \frac{\rho}{A} \Rightarrow Z_c = \frac{\rho c}{A}$

Methods to derive the characteristic impedance, Z_c



$$Z_c = \frac{P}{Q}$$

Definition: Characteristic impedance, Z_c, is the ratio of pulsatile pressure to flow in case of unidirectional waves (no reflections)

1) Direct (definition)

A. Frequency domain: $Z_c = \frac{P(\omega)}{Q(\omega)}$

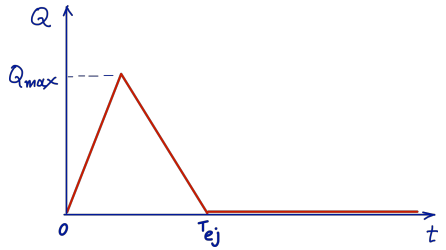
B. Time domain: $Z_c = \frac{\Delta P}{\Delta Q}$

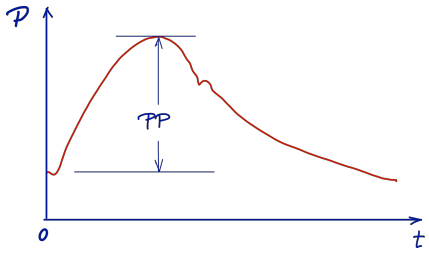
} **Attention: good only for unidirectional waves**

3) From area compliance and geometry: $Z_c = \sqrt{\frac{\rho}{A}\frac{1}{C_A}}$

4) From wave speed and geometry: $Z_c = \frac{\rho c}{A}$

Example: estimate aortic pulse pressure from cardiac output





Assume that aortic flow can be approximated by a triangular wave during the ejection phase, as shown in the Figure. Estimate the **aortic pulse pressure** assuming that:

1. Cardiac output is 5.4 lit/min
2. Heart rate is 72 beat/min
3. Duration of the ejection phase is T_{ej} = 250 ms
4. Ascending aortic diameter is 2.6 cm
5. The wave speed in the ascending aorta is 4.5 m/s

a) The stroke volume is: $Q = SV \cdot HR \Rightarrow SV = \frac{Q}{HR} = \frac{5.4\ lit/min}{72\ beats/min} = 0.075\ lit = \underline{75\ ml}$

b) SV = area under the systolic flow curve $\Rightarrow SV = \frac{1}{2} Q_{max} \cdot T_{ej}$

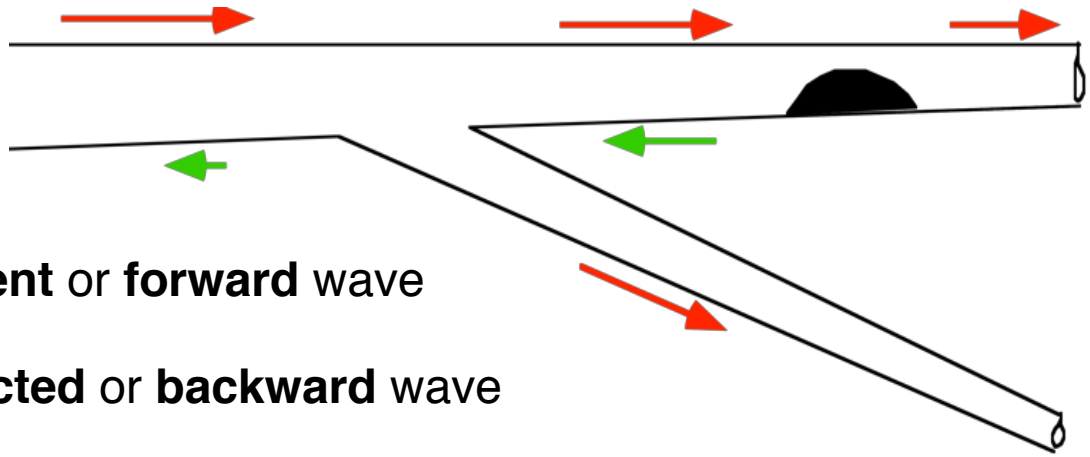
$\Rightarrow Q_{max} = \frac{2 \times SV}{T_{ej}} = \frac{2 \times 75\ ml}{0.250\ s} = \underline{600\ ml/s}$

c) $Z_c = \frac{\rho c}{A} = \frac{1050\ kg/m^3 \cdot 4.5\ m/s}{\pi \cdot 0.013^2\ m^2} = \underline{8.9 \times 10^6\ \frac{kg}{m^4 \cdot s}}$

d) Assuming no wave reflections in early systole: $Z_c = \frac{\Delta P}{\Delta Q} = \frac{PP}{Q_{max}}$

$\Rightarrow PP = Z_c \cdot Q_{max} = 8.9 \times 10^6\ kg/m^4 s \times 600 \times 10^{-6}\ m^3/s = 5340\ N/m^2 \Rightarrow \underline{PP = 40\ mm\ Hg}$

Wave reflections



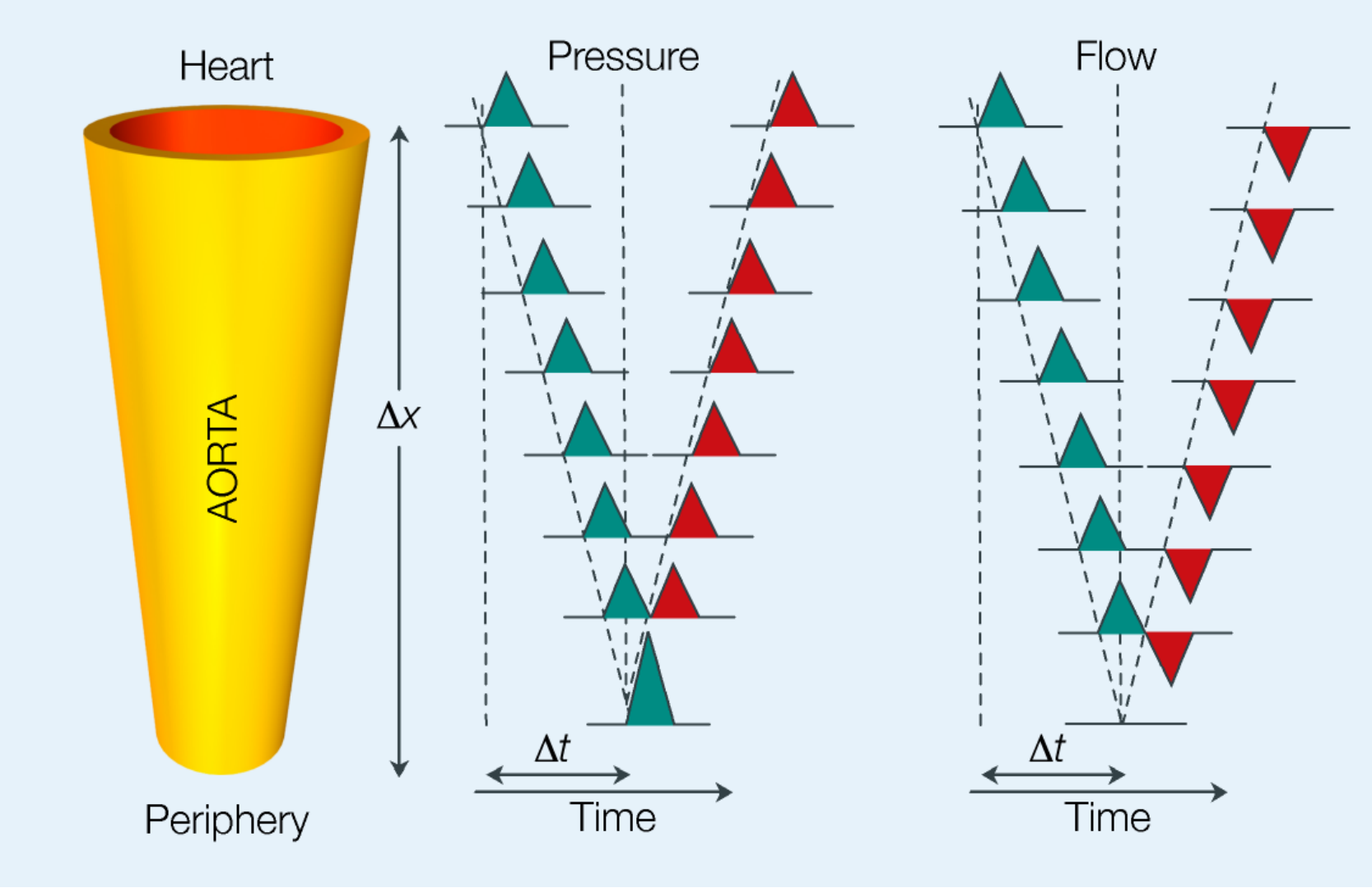
Incident or **forward** wave

Reflected or **backward** wave

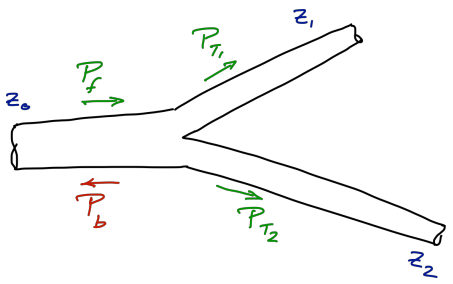
Reflected waves are generated when the transmission properties (characteristic impedance) change abruptly

- bifurcations
- stenoses
- aneurisms
- etc

Wave reflection: schematic interpretation



Wave reflection and transmission at bifurcations



Continuity of pressure at the junction: $P_f + P_b = P_{T_1} = P_{T_2}$ (1)

Continuity of flow at the junction: $Q_f + Q_b = Q_{T_1} + Q_{T_2}$ (2)

$$\Rightarrow \frac{P_f}{z_0} - \frac{P_b}{z_0} = \frac{P_{T_1}}{z_1} + \frac{P_{T_2}}{z_2} \Rightarrow \frac{P_f - P_b}{z_0} = (P_f + P_b)\cdot\left(\frac{1}{z_1} + \frac{1}{z_2}\right)$$

$$\Rightarrow \frac{P_f - P_b}{P_f + P_b} = \frac{z_0}{z_1} + \frac{z_0}{z_2} = \frac{z_0 z_2 + z_0 z_1}{z_1 z_2} \Rightarrow \frac{2P_f}{2P_b} = \frac{z_0 z_1 + z_0 z_2 + z_1 z_2}{z_1 z_2 - z_0 z_1 - z_0 z_2}$$

Reflection coefficient: $R = \frac{P_b}{P_f}$

$$R = \frac{P_b}{P_f} = \frac{z_1 z_2 - z_0 z_1 - z_0 z_2}{z_0 z_1 + z_0 z_2 + z_1 z_2} = \frac{z_0^{-1} - (z_1^{-1} + z_2^{-1})}{z_0^{-1} + z_1^{-1} + z_2^{-1}}$$

Transmission coefficient: $T = \frac{P_{T_1}}{P_f} = \frac{P_{T_2}}{P_f}$

$$T = \frac{P_{T_1}}{P_f} = \frac{P_f + P_b}{P_f} = 1 + \frac{P_b}{P_f} = 1 + R = \frac{2 z_0^{-1}}{z_0^{-1} + z_1^{-1} + z_2^{-1}}$$

The instantaneous power: $\mathcal{P} = F \cdot V = P \cdot A \cdot V = P \cdot Q$

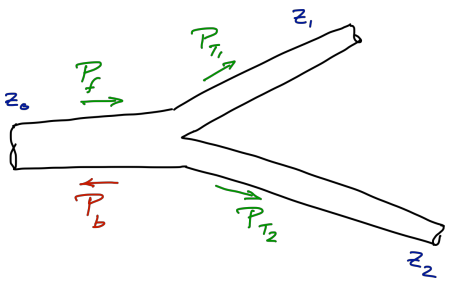
The power of the forward wave is $\mathcal{P}_f = P_f \cdot Q_f = P_f \cdot \frac{P_f}{z_0} = \frac{P_f^2}{z_0}$

The power of the reflected wave: $\mathcal{P}_b = P_b \cdot Q_b = \frac{P_b^2}{z_0} = \frac{(R \cdot P_f)^2}{z_0} = R^2 \cdot \frac{P_f^2}{z_0} = R^2 \cdot \mathcal{P}_f^2$

Power of the transmitted waves: $\mathcal{P}_{T_1} + \mathcal{P}_{T_2} = \frac{P_{T_1}^2}{z_1} + \frac{P_{T_2}^2}{z_2} = \frac{(T P_f)^2}{z_1} + \frac{(T P_f)^2}{z_2}$

$$\frac{\mathcal{P}_{T_1} + \mathcal{P}_{T_2}}{\mathcal{P}_f} = \frac{(T P_f)^2 z_1^{-1} + (T P_f)^2 z_2^{-1}}{P_f^2 z_0^{-1}} = T^2 \frac{z_1^{-1} + z_2^{-1}}{z_0^{-1}}$$

Wave reflection and transmission at bifurcations (2)



Remarks:

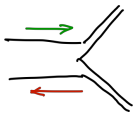
1) Condition for no wave reflections:

$$R = 0 \Rightarrow \frac{1}{Z_0} = \frac{1}{Z_1} + \frac{1}{Z_2} \Rightarrow \frac{A_0}{\rho C_0} = \frac{A_1}{\rho C_1} + \frac{1}{\rho C_2}$$

If $C_0 = C_1 = C_2 \Rightarrow A_0 = A_1 + A_0$

2) Total reflection

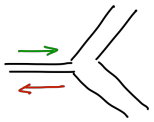
I) $R = 1 \Rightarrow \frac{1}{Z_0} \gg \frac{1}{Z_1} + \frac{1}{Z_2}$



$P_b = P_f \qquad Q_b = -Q_f$

$P = 2P_f \qquad Q = 0$

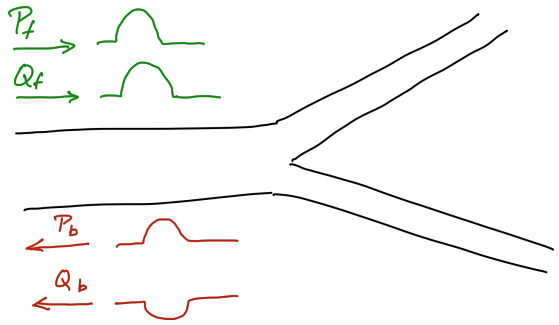
II) $R = -1 \Rightarrow \frac{1}{Z_0} \ll \frac{1}{Z_1} + \frac{1}{Z_2}$



$P_b = -P_f \qquad Q_b = Q_f$

$P = 0 \qquad Q = 2Q_f$

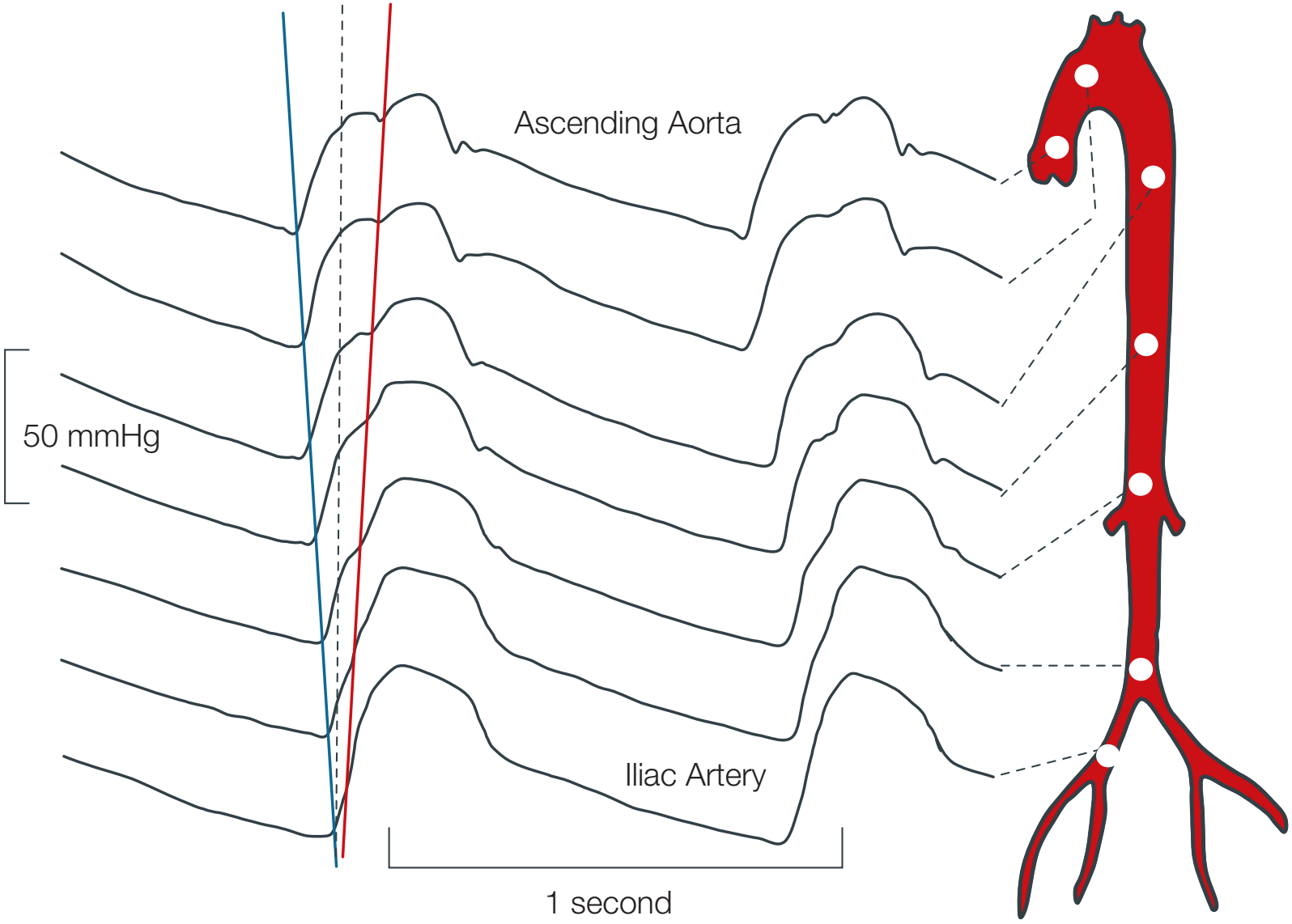
3) In presence of wave reflections, the flow wave is not the same in shape as the pressure wave



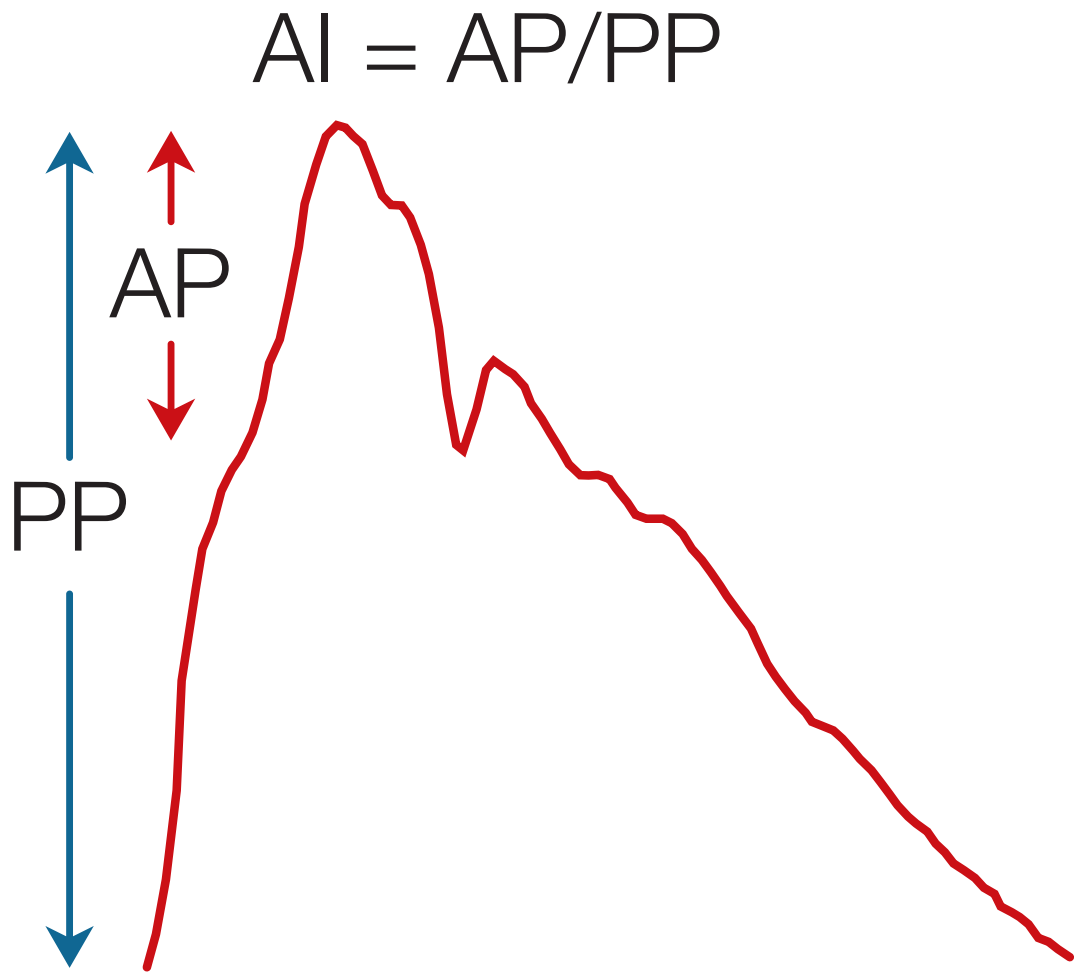
$$P = P_f + P_b = P_f + R P_f = (1+R) P_f$$

$$Q = Q_f + Q_b = \frac{P_f}{Z_0} - \frac{P_b}{Z_0} = \frac{(1-R)}{Z_0} P_f$$

Wave reflection in the aorta: visual interpretation

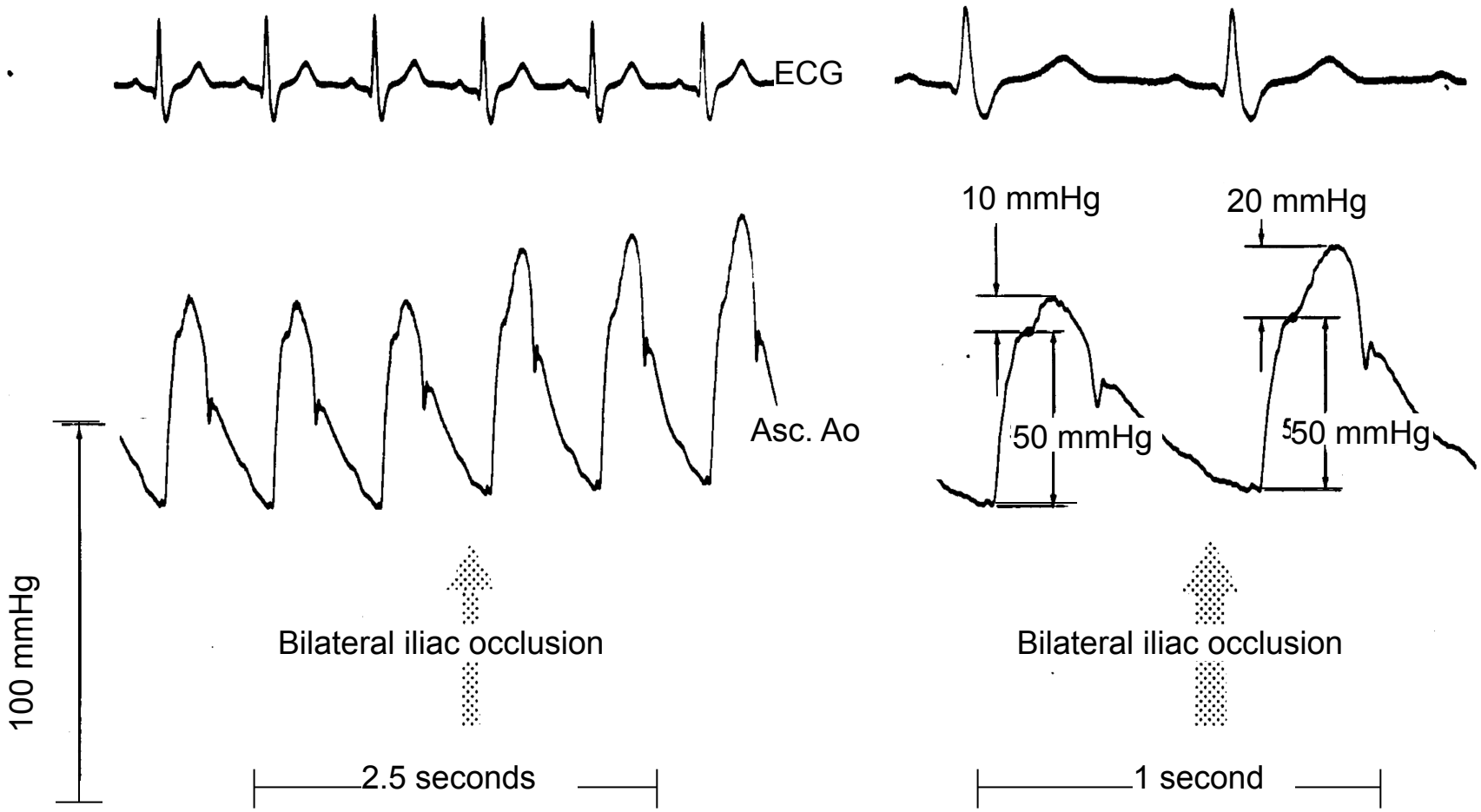


Augmentation index



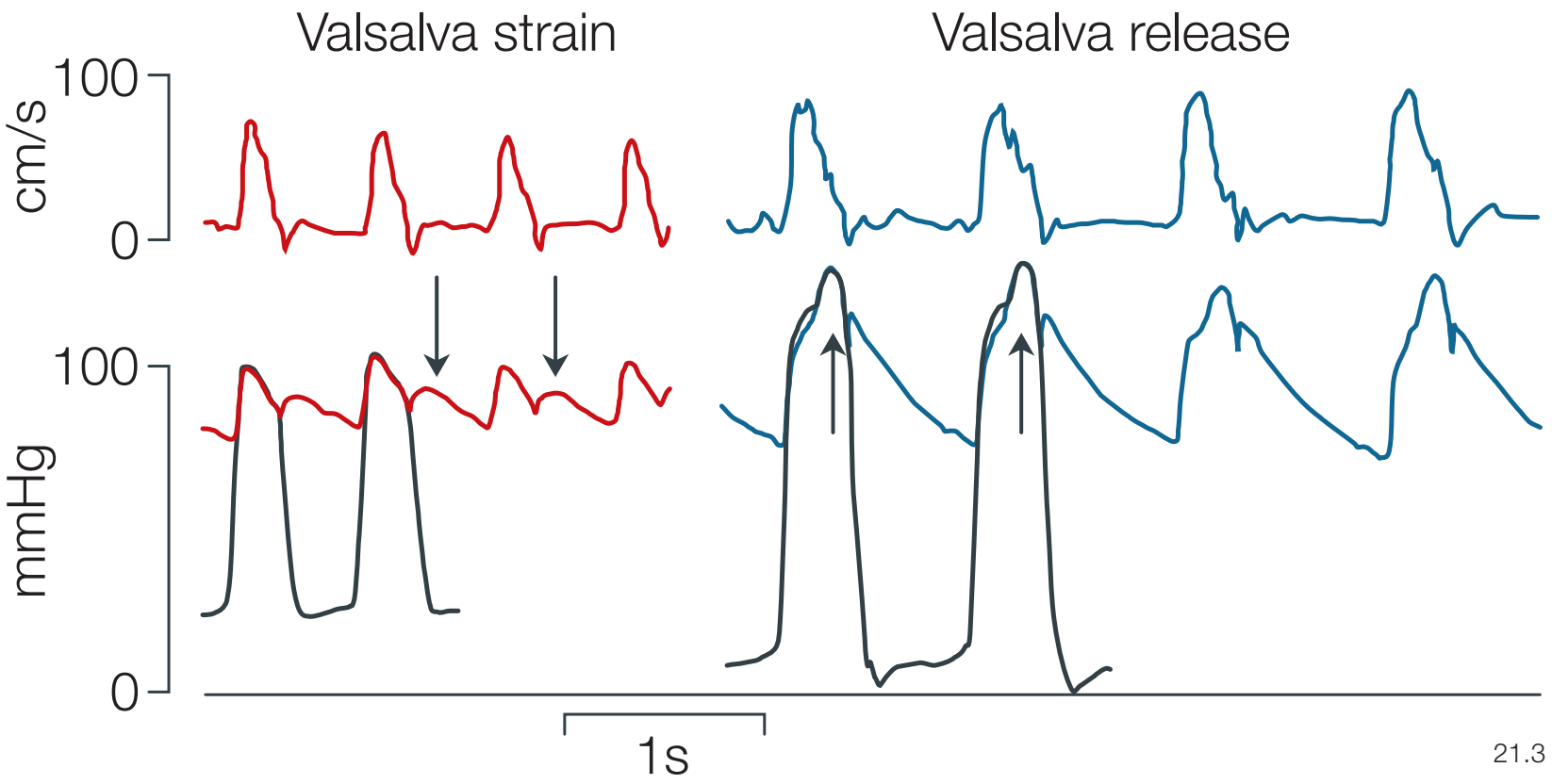
THE AUGMENTATION INDEX (AI) is the augmented pressure (AP) divided by pulse pressure (PP). Calibration of blood pressure is not required.

Wave reflections:
augmentation by peripheral occlusion



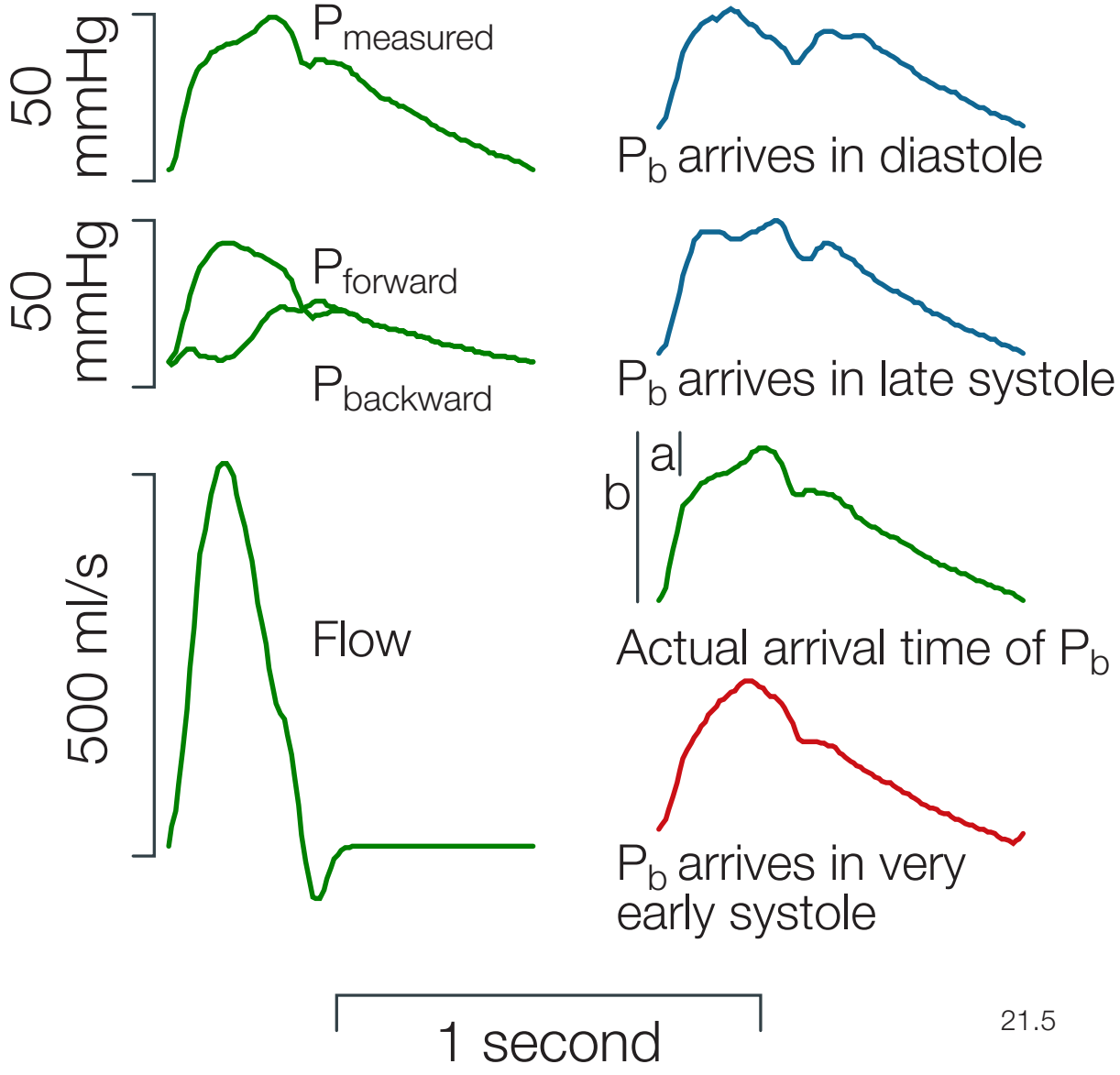
DISTINCT REFLECTION IS INCREASED by mechanical compression of both iliac arteries.

Wave reflections & wave shape

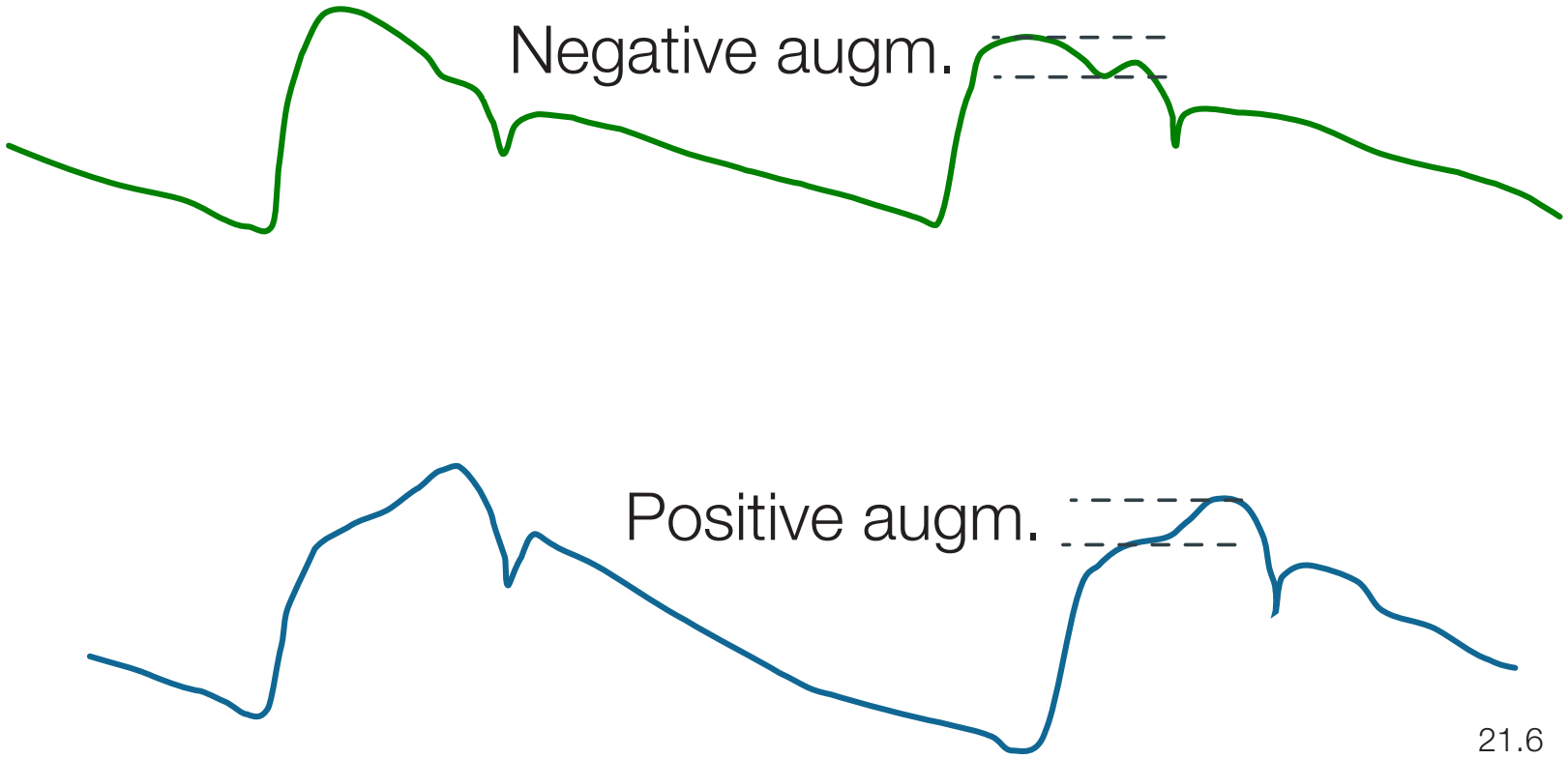


DURING THE VALSALVA strain pressure and flow in systole become alike because diffuse reflections decrease in amplitude and wave speed decreases so that reflections return in diastole.

Wave reflections & wave shape



Wave reflections & wave shape



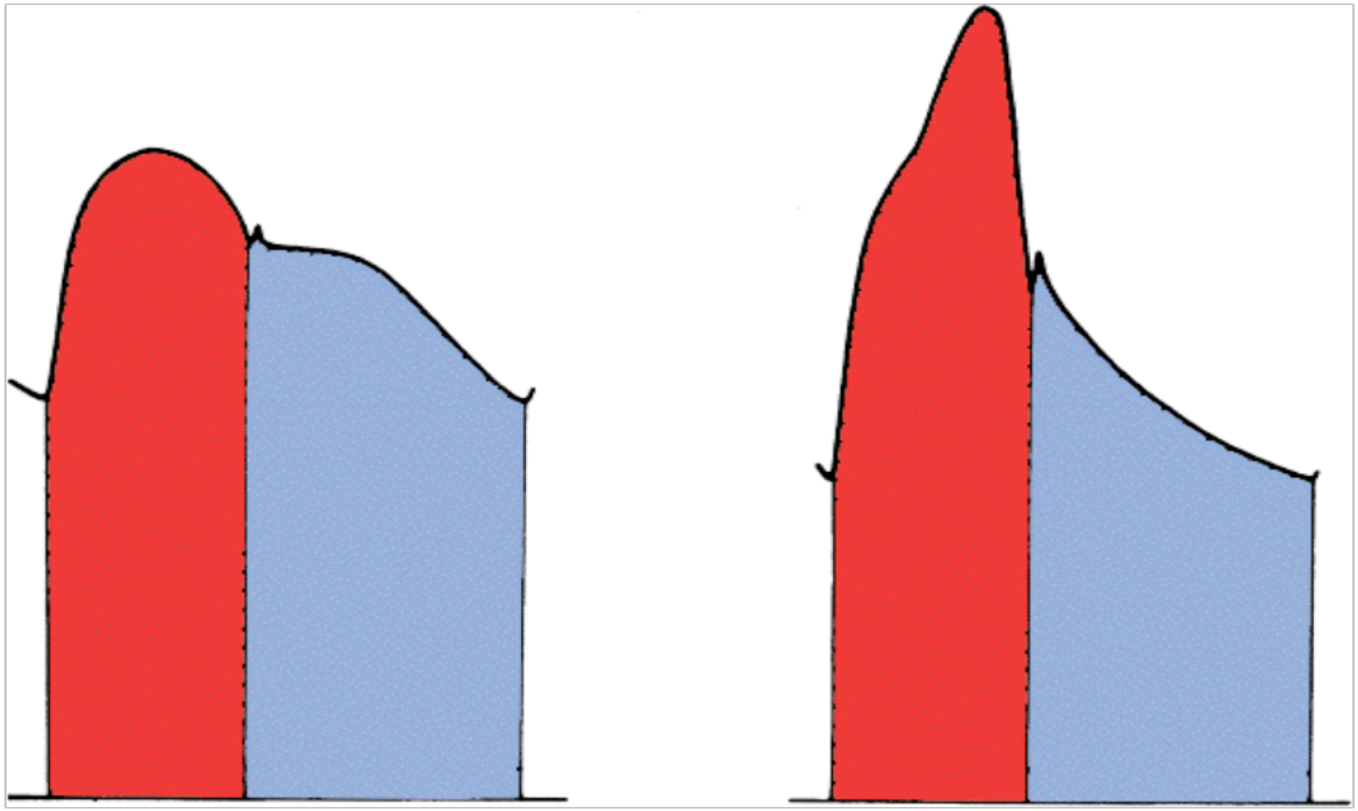
Aortic pressures in young and old subjects

Young

Type C

Old

Type A



Fourier analysis of arterial waves

Assume quasi-linear arterial system:
Pressure and flow pulses can be broken down to their Fourier components

⇓

Treat each harmonic separately

⇓

Superpose all harmonics to get the result

$$P(t) = P_0 + \sum_{n=1}^{N} P_n \cos(n\omega t + \phi_n)$$

P_0: mean value

P_n: modulus

ϕ_n: phase

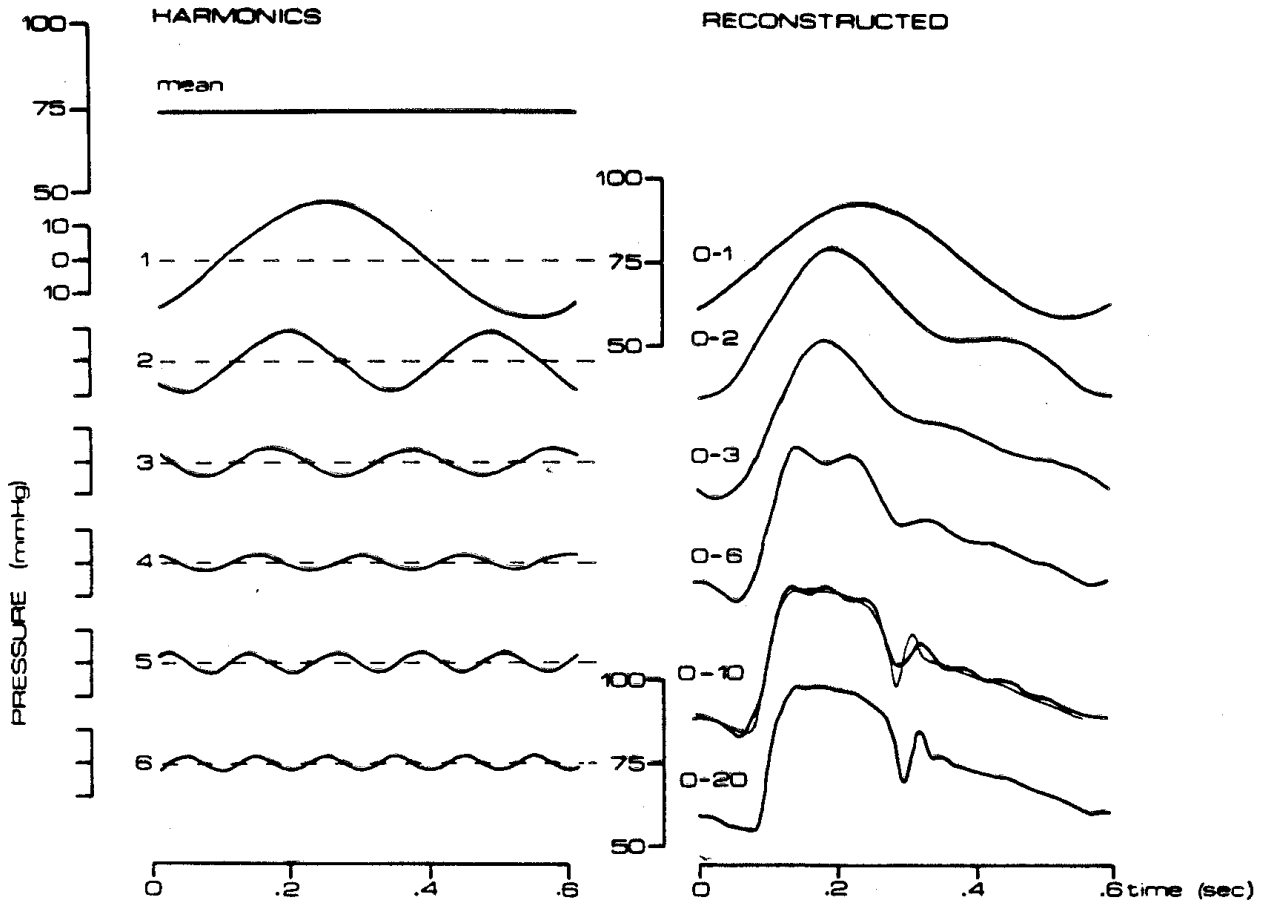
$n = 1 \ldots N$: harmonics

Or, in complex form

$$P(t) = P_0 + \sum_{n=1}^{N} P_n e^{i(n\omega t + \phi_n)}$$

(only the real part)

Fourier analysis of arterial waves



In general, 10 to 20 harmonics are enough to get an accurate representation of the pressure and flow wave

Input impedance

$Z_{in} = P/Q$

Definition: Input impedance is the ratio of Pressure *(P)* to flow *(Q)* expressed in the Fourier domain

- Z_{in} is defined in presence or absence of reflections
- In case of no reflections, $Z_{in} = Z_c$

Input impedance is defined only in the Fourier domain:

$$P = P_0 + \sum_{n=1}^{N} P_n e^{i(n\omega t + \phi_n)}$$

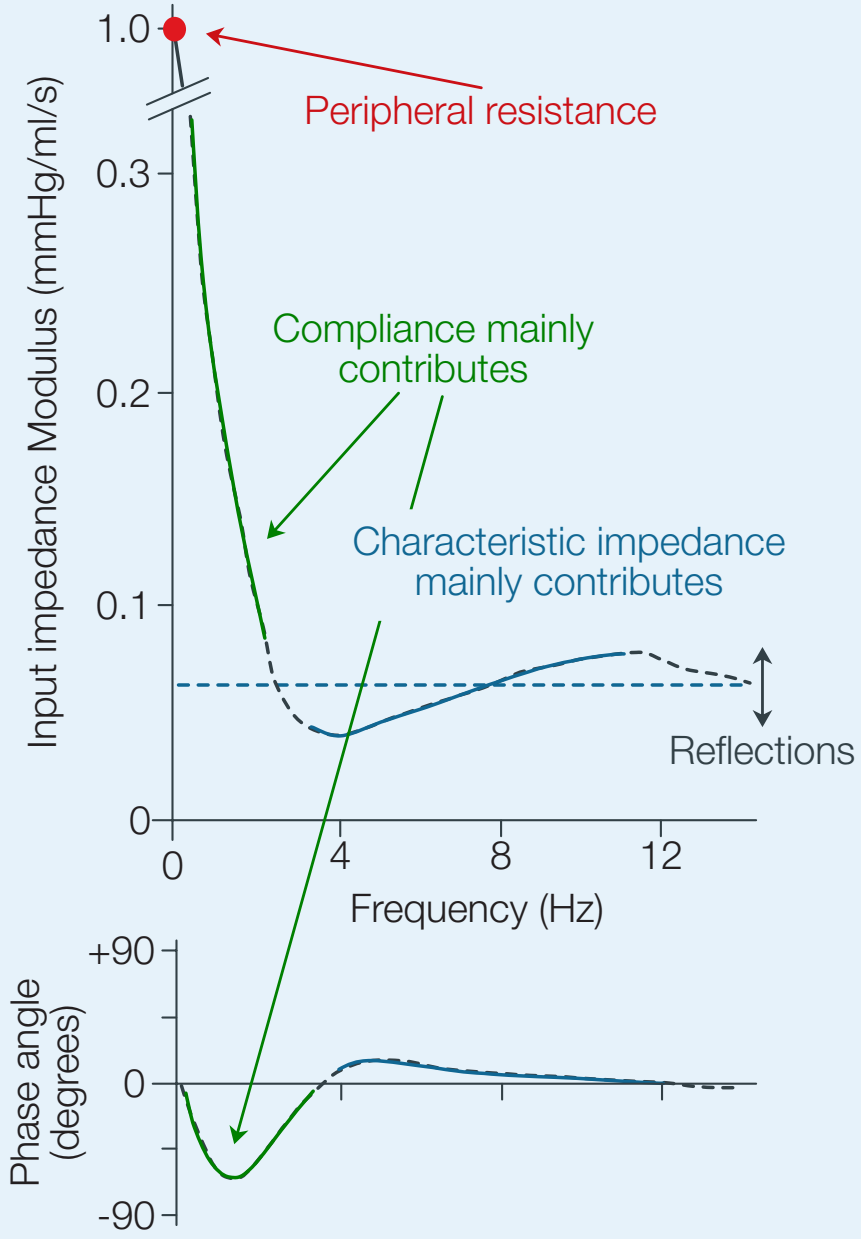
$$Q = Q_0 + \sum_{n=1}^{N} Q_n e^{i(n\omega t + \theta_n)}$$

$$Z_{in} = \begin{cases} \frac{P_0}{Q_0} & \text{mean value } (0^{th} \text{ harmonic}) \\ \frac{P_n}{Q_n} e^{i(n\omega t + (\phi_n - \theta_n))} & n^{th} \text{ harmonic} \end{cases}$$

$(\phi_n - \theta_n)$: phase

$\frac{P_n}{Q_n}$: amplitude

Input impedance



$$Z_{in} = P/Q$$

Pressure (P) and flow (Q) expressed in the Fourier domain

Reflection coefficient at a specific arterial location

Reflection coefficient: $\Gamma = \frac{P_b}{P_f}$

$$P = P_f + P_b$$

$$Q = Q_f + Q_b \Rightarrow \frac{P}{Z_{in}} = \frac{P_f}{Z_c} - \frac{P_b}{Z_c} \Rightarrow$$

$$Z_{in} = Z_c \cdot \frac{P_f + P_b}{P_f - P_b} \Rightarrow \frac{Z_{in}}{Z_c} = \frac{P_f + P_b}{P_f - P_b}$$

$$\Rightarrow \frac{Z_{in} - Z_c}{Z_{in} + Z_c} = \frac{2P_b}{2P_f} = \frac{P_b}{P_f} = \Gamma$$

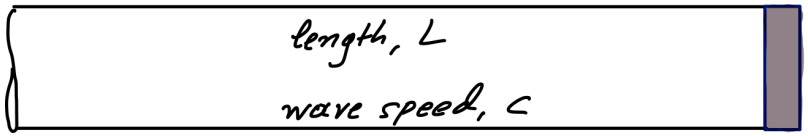
$$\therefore \Gamma = \frac{Z_{in} - Z_c}{Z_{in} + Z_c}$$

So, when $\Gamma = 0$ (no reflections) $Z_{in} = Z_c$

Example: input impedance in a frictionless tube with a fully blocked end

$P_f = P_0 \sin \omega t$ → $Q_f = \frac{P_0}{Z_c} \sin \omega t$

$P_f = P_0 \sin\left[\omega\left(t - \frac{L}{c}\right)\right]$ → $Q_f = \frac{P_0}{Z_c} \sin\left[\omega\left(t - \frac{L}{c}\right)\right]$



$\Gamma = 1$ (total reflection)

$P_b = P_0 \sin\left[\omega\left(t - \frac{2L}{c}\right)\right]$ ← $Q_b = -\frac{P_0}{Z_c} \sin\left[\omega\left(t - \frac{2L}{c}\right)\right]$

$P_b = P_0 \sin\left[\omega\left(t - \frac{L}{c}\right)\right]$ ← $Q_b = -\frac{P_0}{Z_c} \sin\left[\omega\left(t - \frac{L}{c}\right)\right]$

Z_{in} at the entry of the tube:

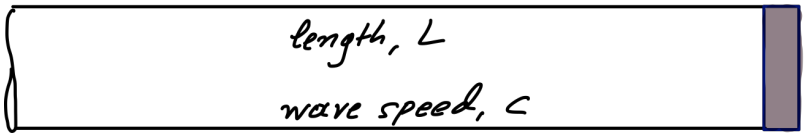
$$Z_{in} = \frac{P}{Q} = \frac{P_f + P_b}{Q_f + Q_b} = \frac{P_0\left[\sin\omega t + \sin\left[\omega\left(t - \frac{2L}{c}\right)\right]\right]}{\frac{P_0}{Z_c}\left[\sin\omega t - \sin\left[\omega\left(t - \frac{2L}{c}\right)\right]\right]}$$

$$\Rightarrow Z_{in} = Z_c \frac{\sin\omega t + \sin\left[\omega\left(t - \frac{2L}{c}\right)\right]}{\sin\omega t - \sin\left[\omega\left(t - \frac{2L}{c}\right)\right]}$$

If reflection were partial $\Gamma < 1$ and not total, we would have:

$$Z_{in} = Z_c \frac{\sin\omega t + \Gamma \sin\left[\omega\left(t - \frac{2L}{c}\right)\right]}{\sin\omega t - \Gamma \sin\left[\omega\left(t - \frac{2L}{c}\right)\right]}$$

Example: input impedance in a frictionless tube with a fully blocked end



$$Z_{in} = Z_c \frac{\sin\omega t + \Gamma \sin\left[\omega\left(t - \frac{2L}{c}\right)\right]}{\sin\omega t - \Gamma \sin\left[\omega\left(t - \frac{2L}{c}\right)\right]}$$

Z_{in} is zero ($\Gamma = 1$) or minimum ($\Gamma < 1$) when $\omega \frac{2L}{c} = \pi \Rightarrow 2\pi f \cdot \frac{2L}{c} = \pi$

$$\Rightarrow \quad L = \frac{1}{4}\frac{c}{f} = \frac{1}{4}\lambda$$

The quarter wavelength formula

Remarks:

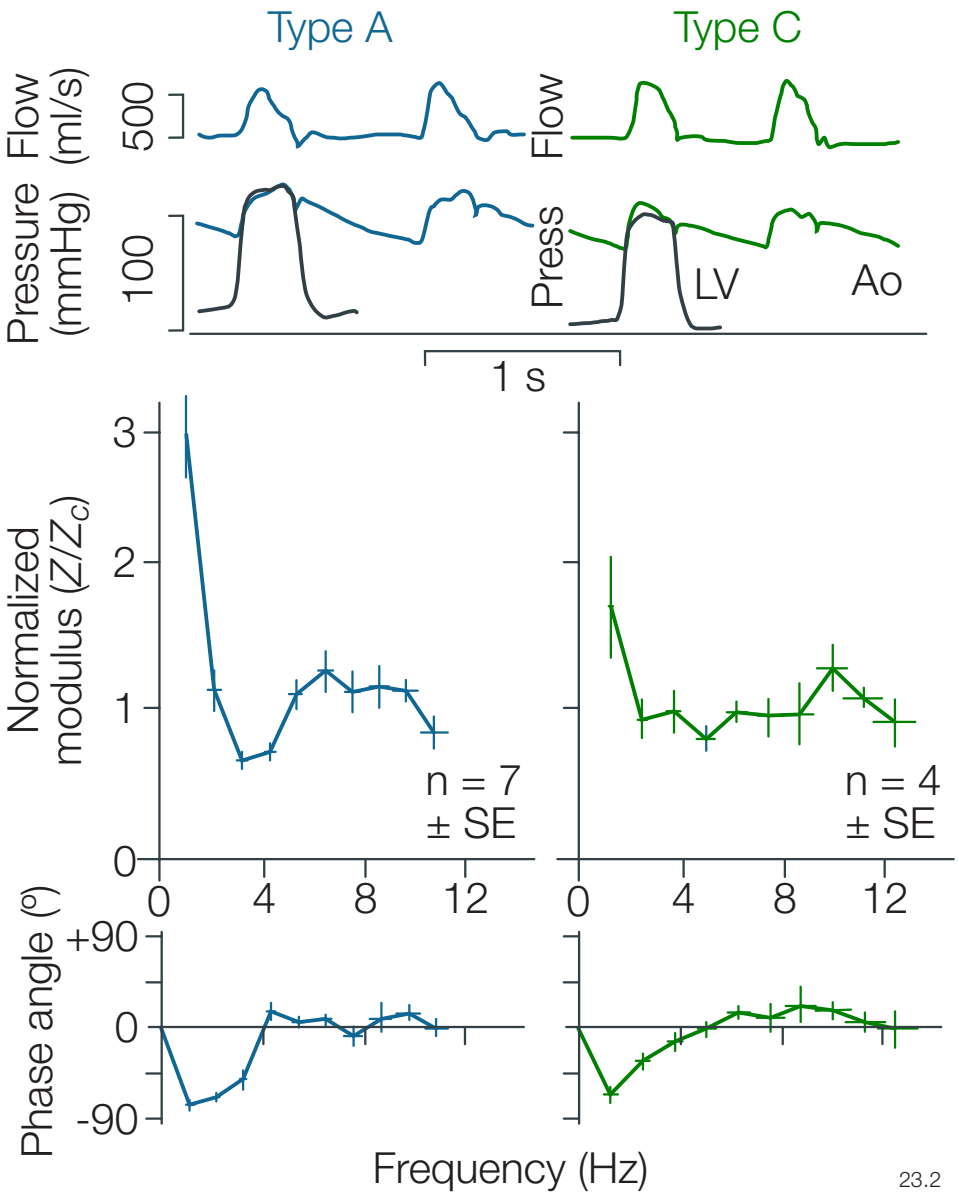
1) The first minimum in $|Z_{in}|$ indicates the "effective length" of the arterial system

Example: $|Z_{in}|$ minimum @ 3.8 Hz

$$L_{eff} = \frac{1}{4}\frac{c}{f} = \frac{1}{4}\cdot\frac{5\ m/s}{3.8\ Hz} = 0.33\ m$$

2) When $|Z_{in}|$ is at its first minimum the phase of Z_{in} is zero (zero crossover from negative to positive phase values)

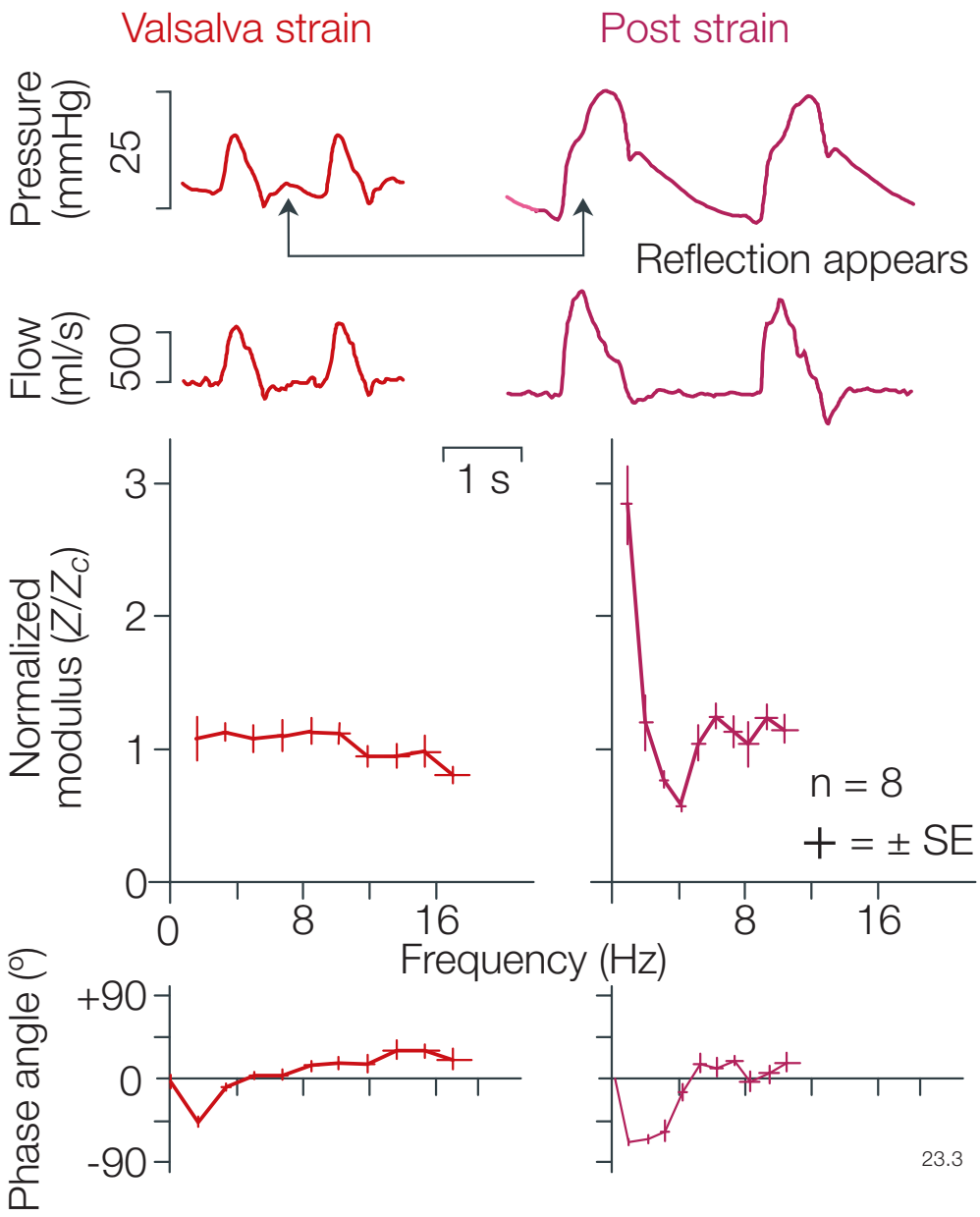
Input impedance in the young (type C) and the aged (type A)



23.2

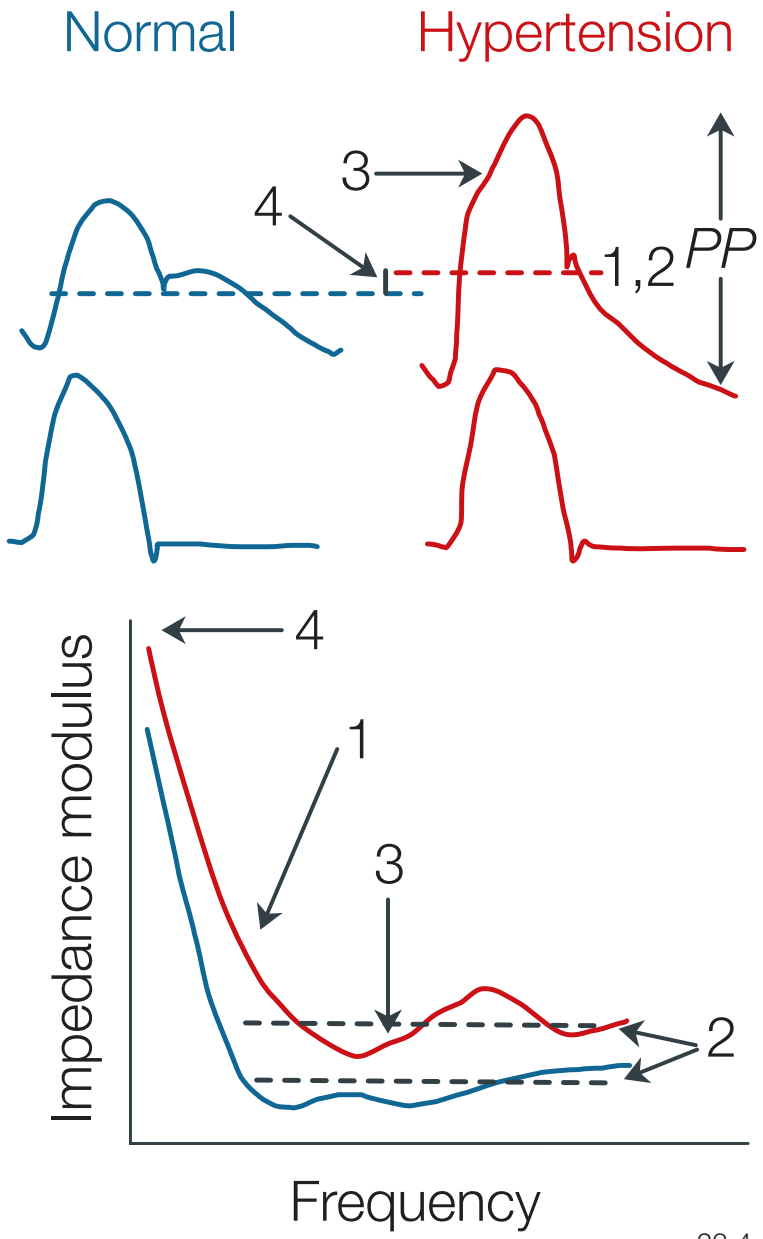
TYPES OF BEATS RELATE TO INPUT IMPEDANCE. In older subjects, Type A, with high pulse wave velocity, reflections return in systole and augment the pressure wave. The impedance oscillates about the characteristic impedance. In young subjects, Type C, reflections are smaller and return in diastole. The impedance oscillates less.

Input impedance



VALSALVA STRAIN increases thoracic and abdominal pressures. The lower transmural pressure increases arterial compliance and lowers pulse wave velocity. Reflections diminish and return later, in diastole. An almost reflectionless situation appears where pressure and flow resemble each other and input impedance equals aortic characteristic impedance. In the release phase the reverse is true, reflections return in systole and are large.

Effects of hypertension



1. Higher peripheral resistance
2. Lower compliance
3. Higher wave speed
4. Higher characteristic impedance

IN HYPERTENSION peripheral resistance and thus mean pressure is increased (1). Compliance is decreased resulting in a less rapid decrease in impedance, with frequency (2) and a higher characteristic impedance (4). Pulse pressure, PP, increases (2). Wave speed is increased, the impedance oscillates more around the characteristic impedance, the first impedance modulus moves to higher frequencies (3) and the wave is augmented (3).

Input impedance examples: hypertension (I)

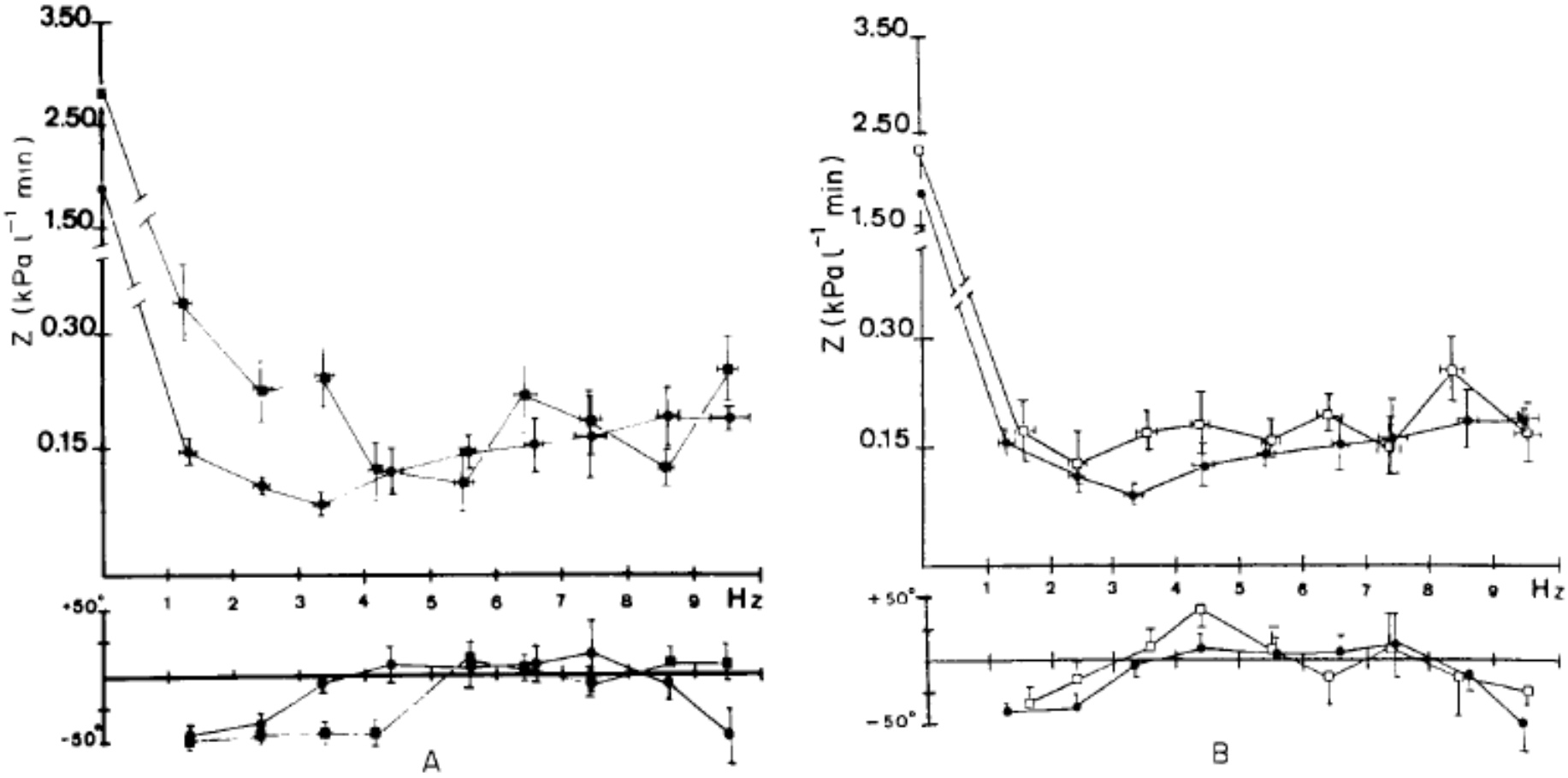
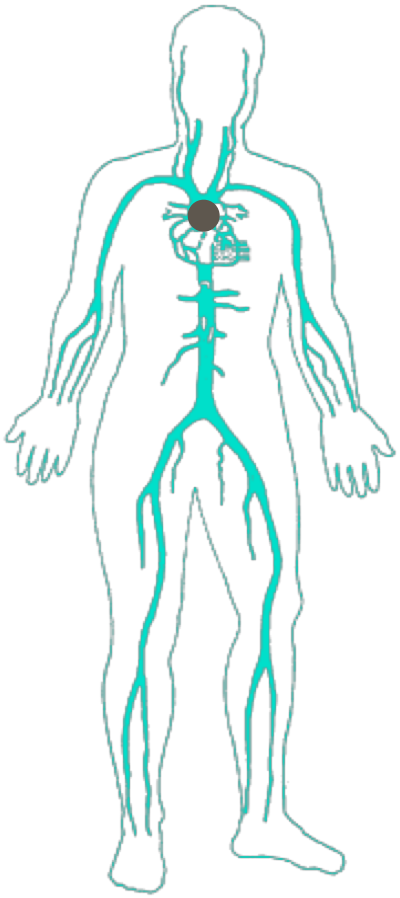
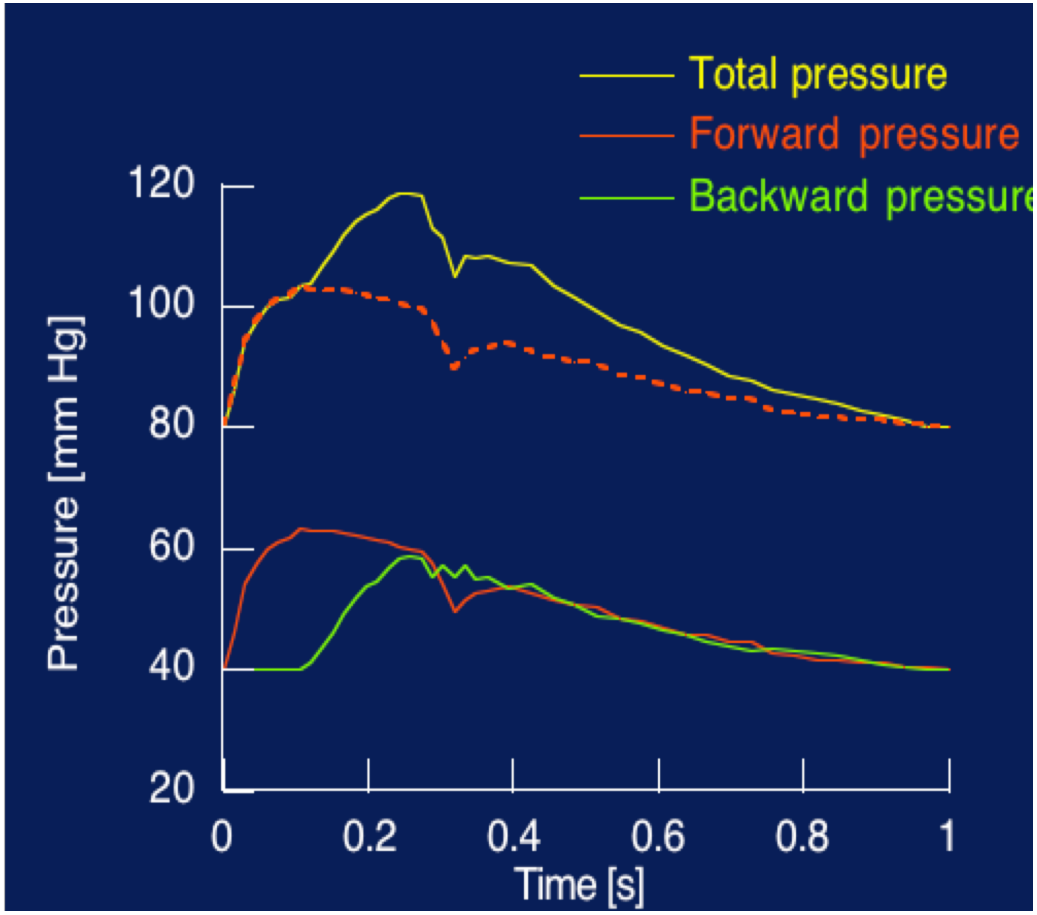


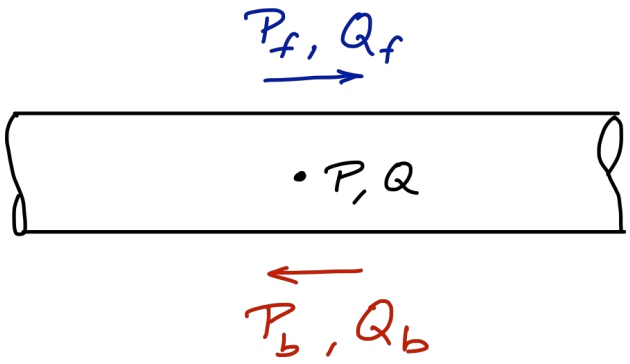
Fig. 16.13 (A) Impedance modulus and phase in a group of normal subjects (closed circles) and in a group of hypertensive patients of similar age (closed squares). **(B)** Change in ascending aortic impedance of hypertensive patients with infusion of nitroprusside to restore normal mean arterial pressure (open circles). Closed circles represent data from normal subjects as in (A). Reproduced from Merillon et al. (1982).

Wave analysis





Separation of waves into forward and backward running components



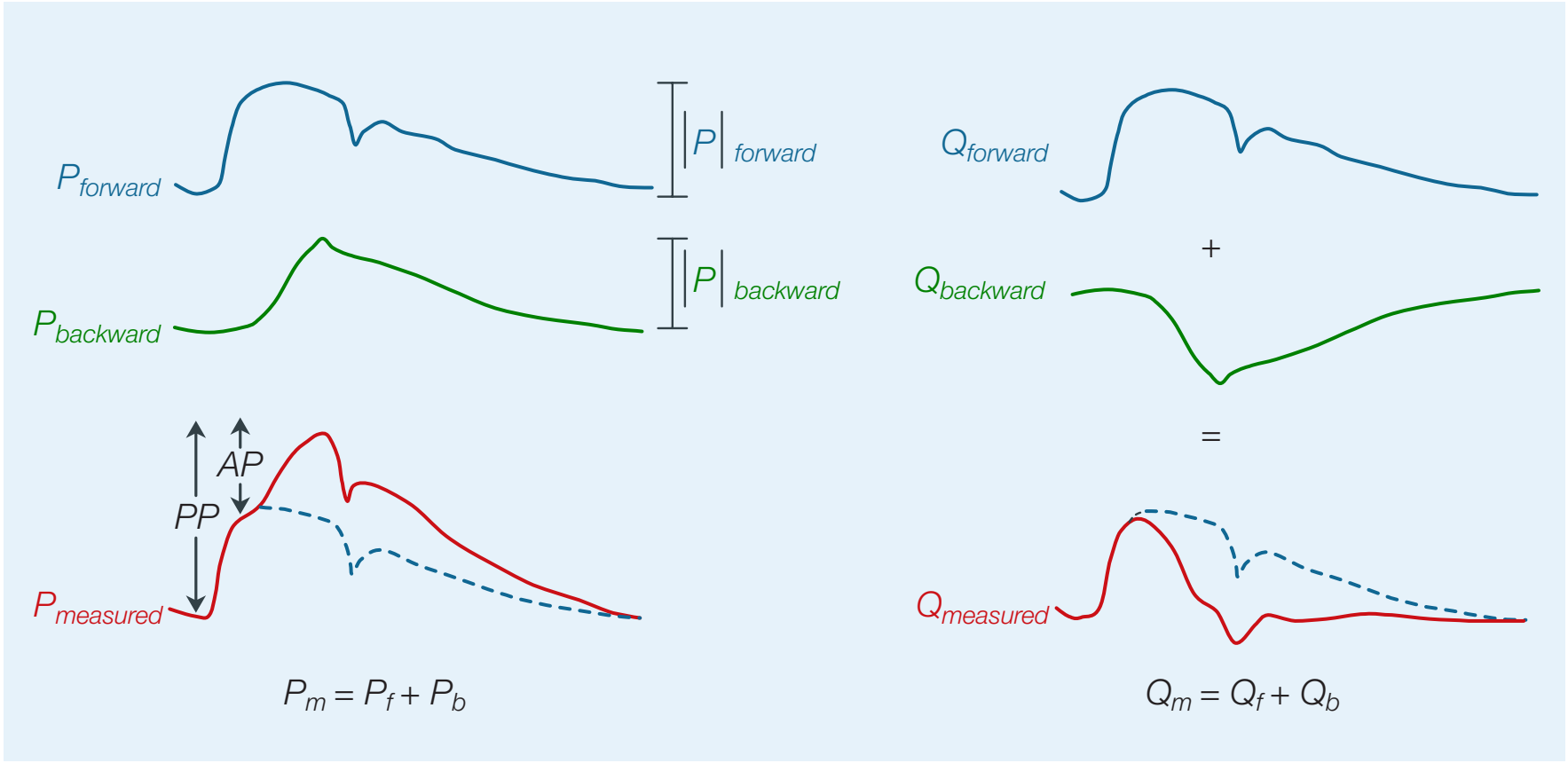
$$P = P_f + P_b \qquad (1)$$

$$Q = Q_f + Q_b = \frac{P_f}{Z_c} - \frac{P_b}{Z_c} \Rightarrow Z_c \cdot Q = P_f - P_b \qquad (2)$$

$$(1) + (2) \Rightarrow P + Z_c Q = 2P_f \Rightarrow \underline{P_f = \frac{P + Z_c Q}{2}}$$

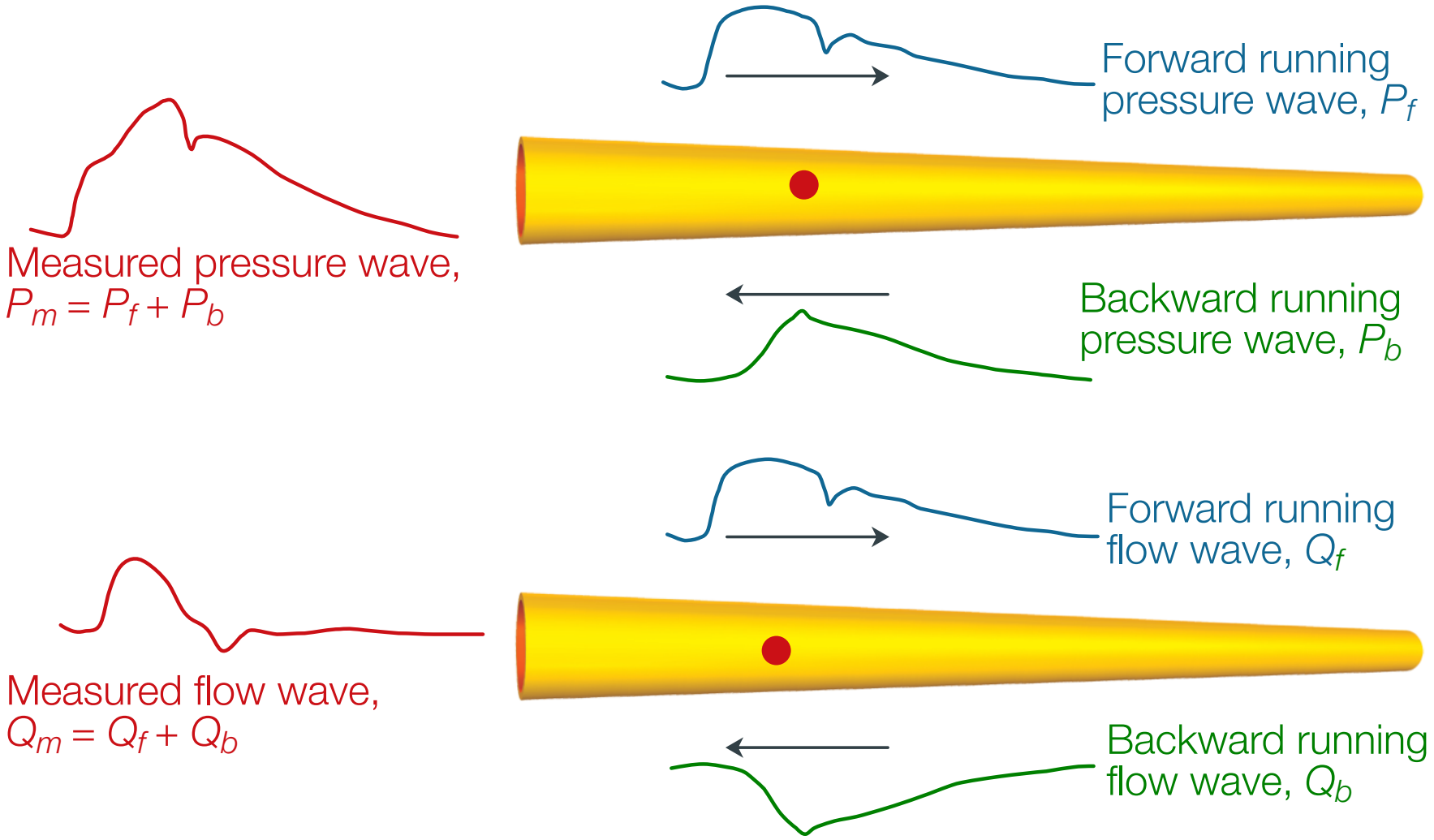
$$(1) - (2) \Rightarrow P - Z_c Q = 2P_b \Rightarrow \underline{P_b = \frac{P - Z_c Q}{2}}$$

Separation of waves into forward and backward running components

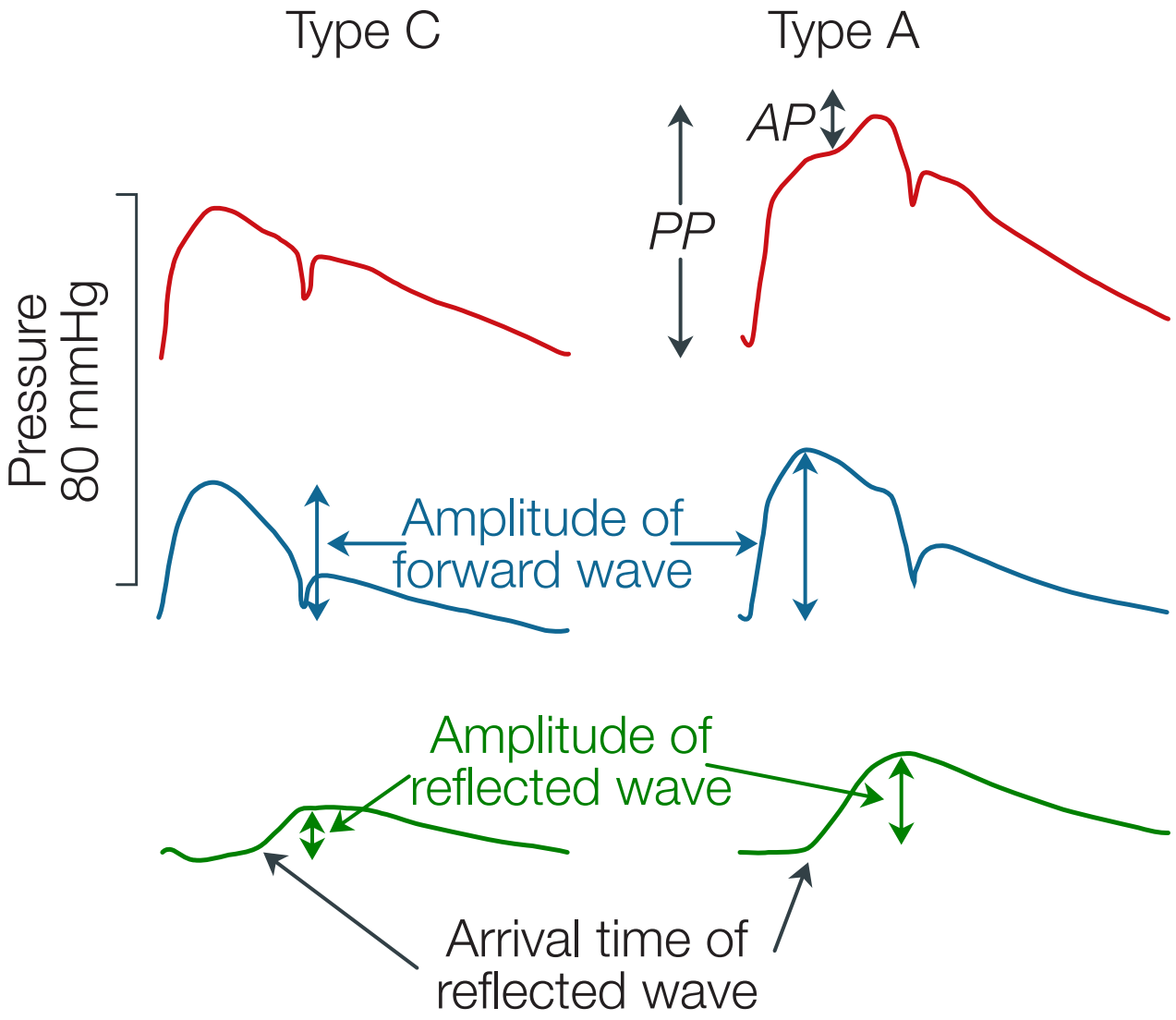


$$P_f = \frac{P + Z_c \cdot Q}{2} \qquad P_b = \frac{P - Z_c \cdot Q}{2}$$

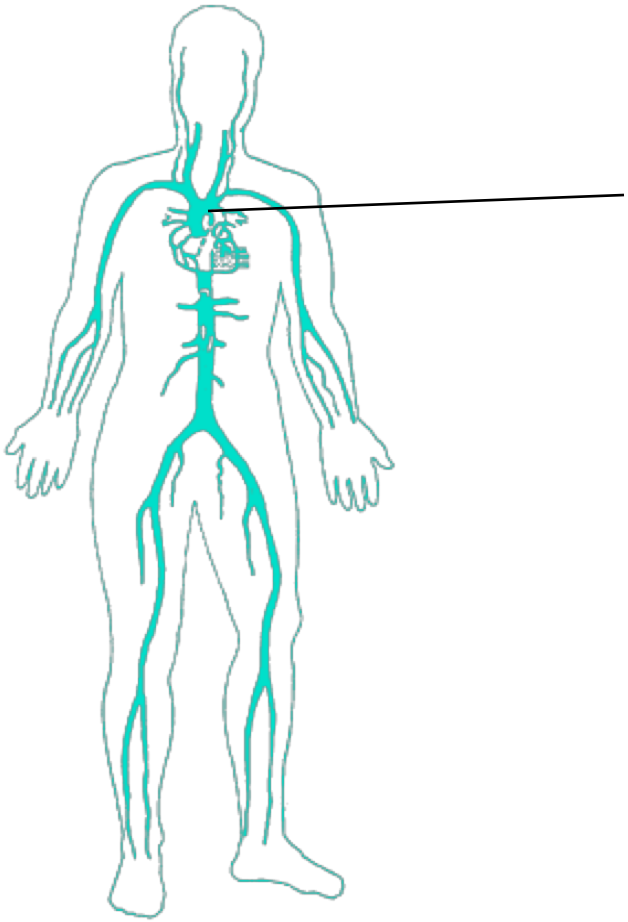
Separation of waves into forward and backward running components

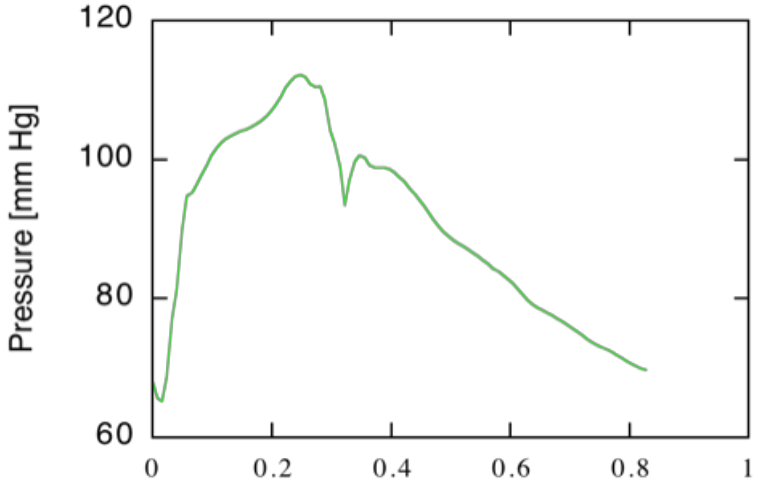


Analysis of arterial waves



Aortic pressure

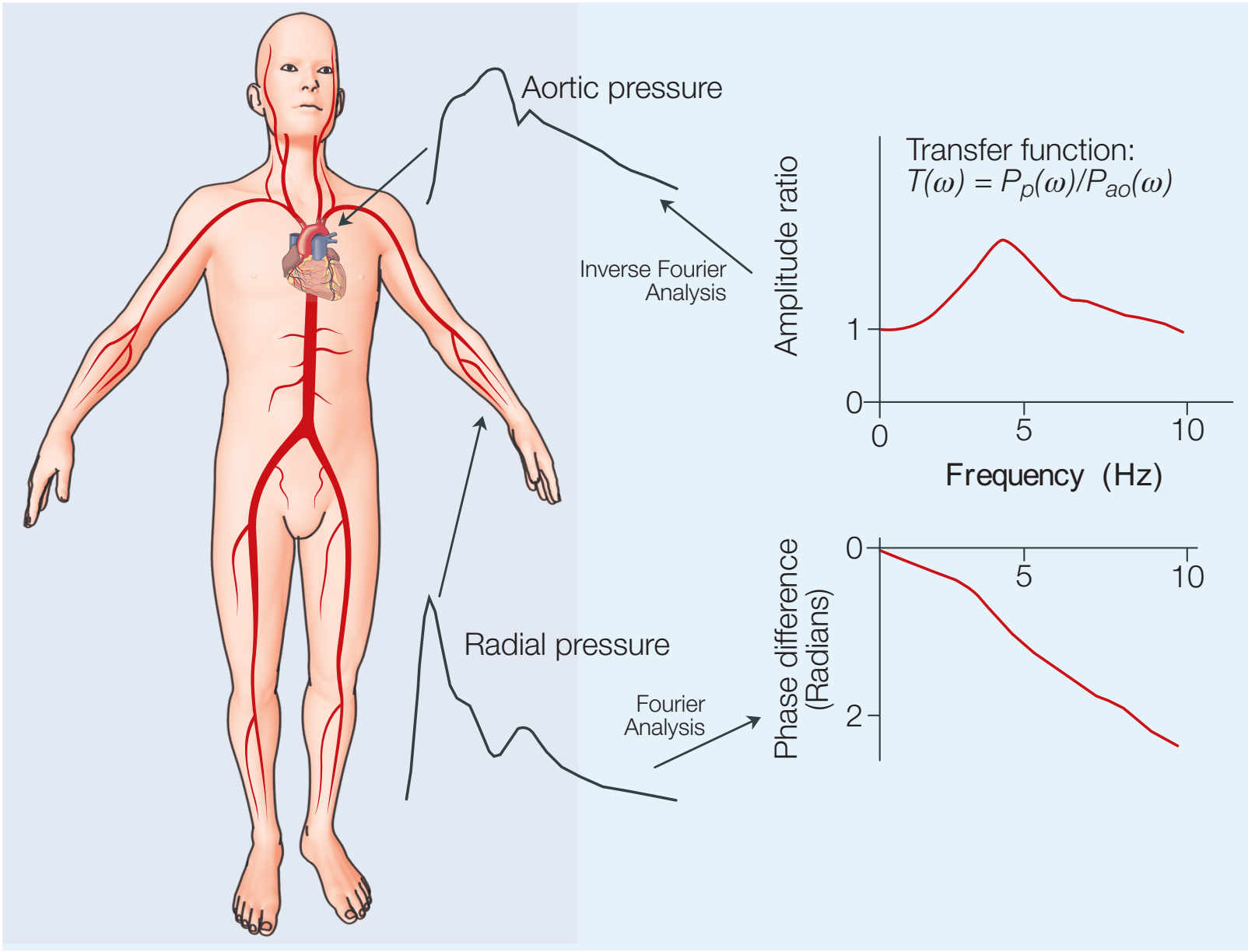




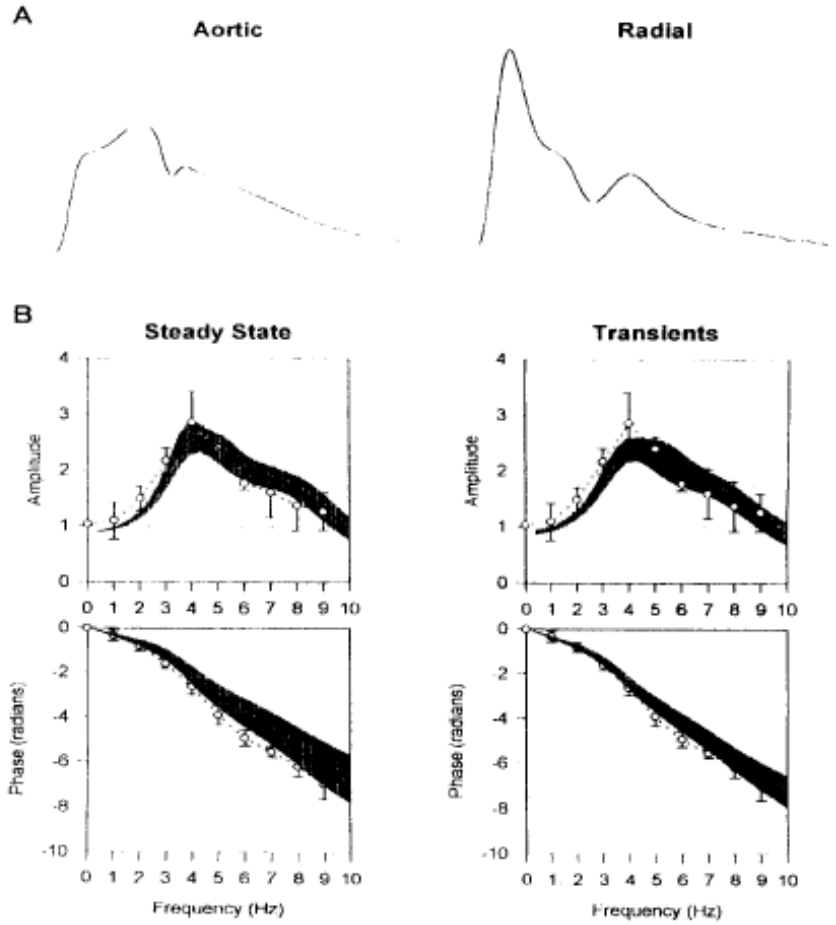
Aortic pressure defines/used for:

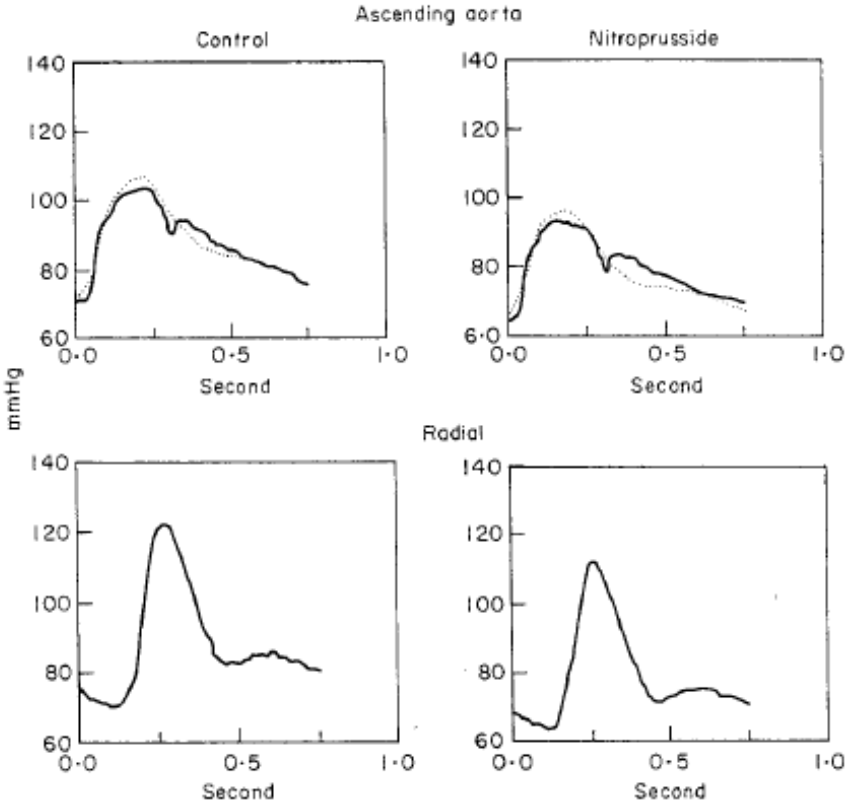
- Cardiac afterload
- Augmentation index
- Compliance estimation
- Systolic & diastolic pressure

Transfer function



Pressure transfer from periphery to the aorta





Not satisfactory!!!

Transfer function along a single arterial segment

Γ = reflection coef. @ distal end

Let $\gamma = \alpha + i\beta = \frac{i\omega}{c}$ be the complex propagation coefficient

@ Proximal end

$$P_f = P_0 e^{i\omega t}$$

$$Q_f = \frac{P_0}{Z_c} e^{i\omega t}$$

@ Distal end

$$P_f = P_0 e^{i\omega t} e^{-\gamma L}$$

$$Q_f = \frac{P_0}{Z_c} e^{i\omega t} e^{-\gamma L}$$

$$P_b = \Gamma P_0 e^{i\omega t} e^{-2\gamma L}$$

$$Q_b = -\Gamma \frac{P_0}{Z_c} e^{i\omega t} e^{-2\gamma L}$$

$$P_b = \Gamma P_0 e^{i\omega t} e^{-\gamma L}$$

$$Q_b = -\Gamma \frac{P_0}{Z_c} e^{i\omega t} e^{-\gamma L}$$

$$Z_{in} = \frac{P_p}{Q_p} = \frac{P_0 e^{i\omega t} + \Gamma P_0 e^{i\omega t} e^{-2\gamma L}}{\frac{P_0}{Z_c} e^{i\omega t} - \Gamma \frac{P_0}{Z_c} e^{i\omega t} e^{-2\gamma L}}$$

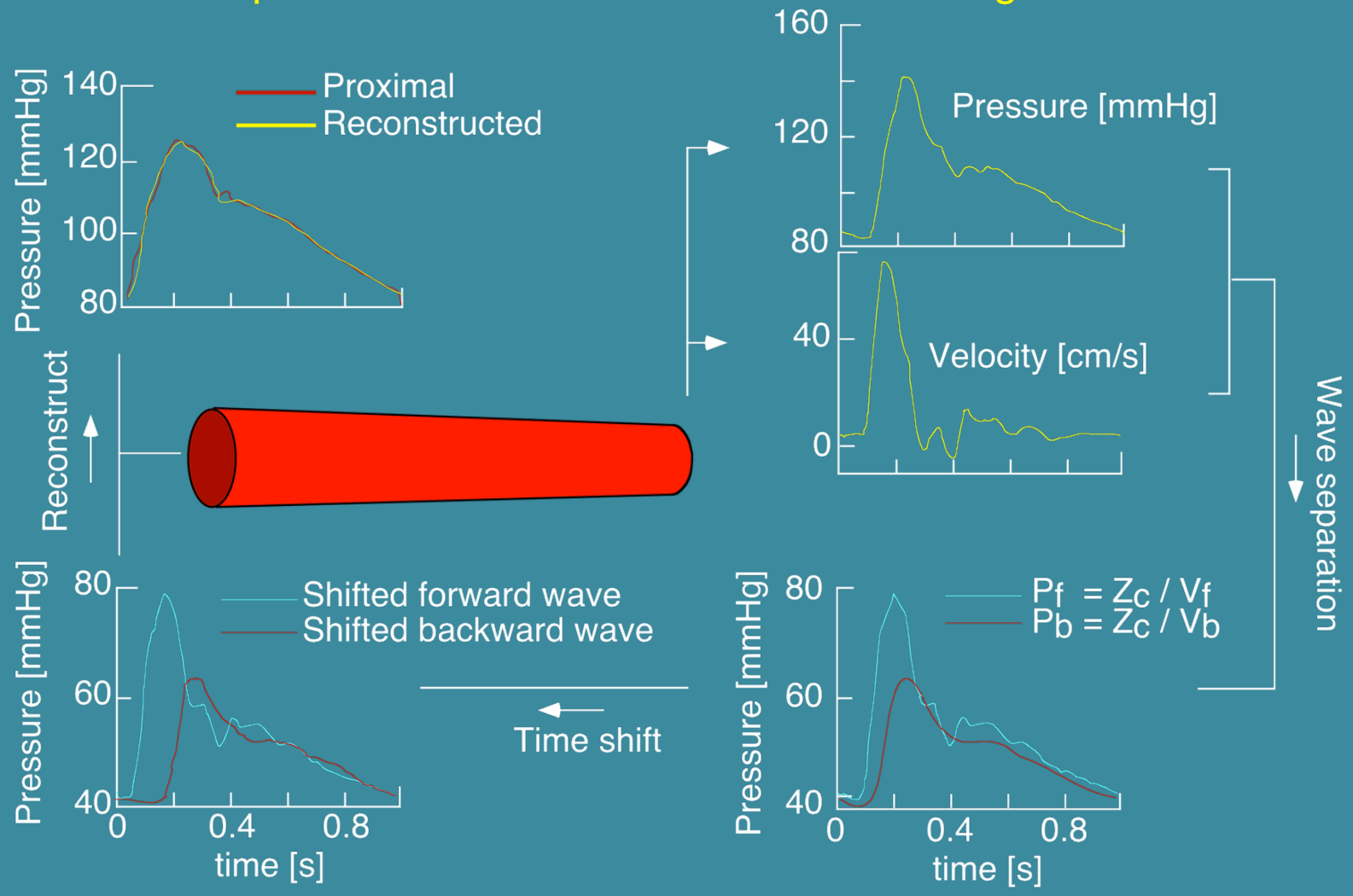
$$\Rightarrow \quad Z_{in} = Z_c \cdot \frac{1 + \Gamma e^{-2\gamma L}}{1 - \Gamma e^{-2\gamma L}}$$

$$T_P = \frac{P_d}{P_p} = \frac{P_0 e^{i\omega t} e^{-\gamma L} + \Gamma P_0 e^{i\omega t} e^{-\gamma L}}{P_0 e^{i\omega t} + \Gamma P_0 e^{i\omega t} e^{-2\gamma L}} \Rightarrow \quad T_P = \frac{1 + \Gamma}{e^{\gamma L} + \Gamma e^{-\gamma L}}$$

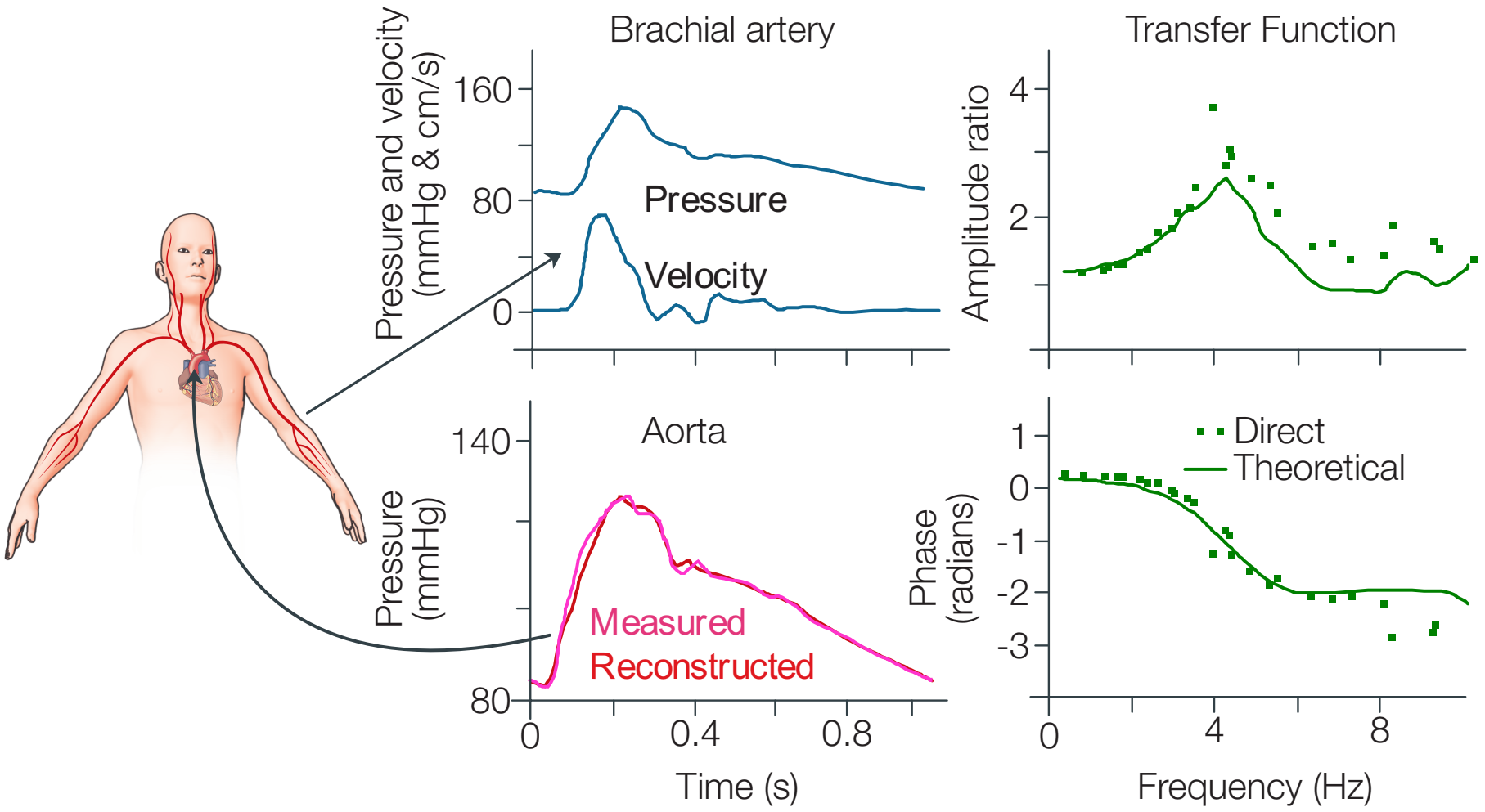
$$T_Q = \frac{Q_d}{Q_f} = \frac{\frac{P_0}{Z_c} e^{i\omega t} e^{-\gamma L} - \Gamma \frac{P_0}{Z_c} e^{i\omega t} e^{-\gamma L}}{\frac{P_0}{Z_c} e^{i\omega t} - \Gamma \frac{P_0}{Z_c} e^{i\omega t} e^{-2\gamma L}} \Rightarrow \quad T_Q = \frac{1 - \Gamma}{e^{\gamma L} - \Gamma e^{-\gamma L}}$$

Physical basis of pressure transfer

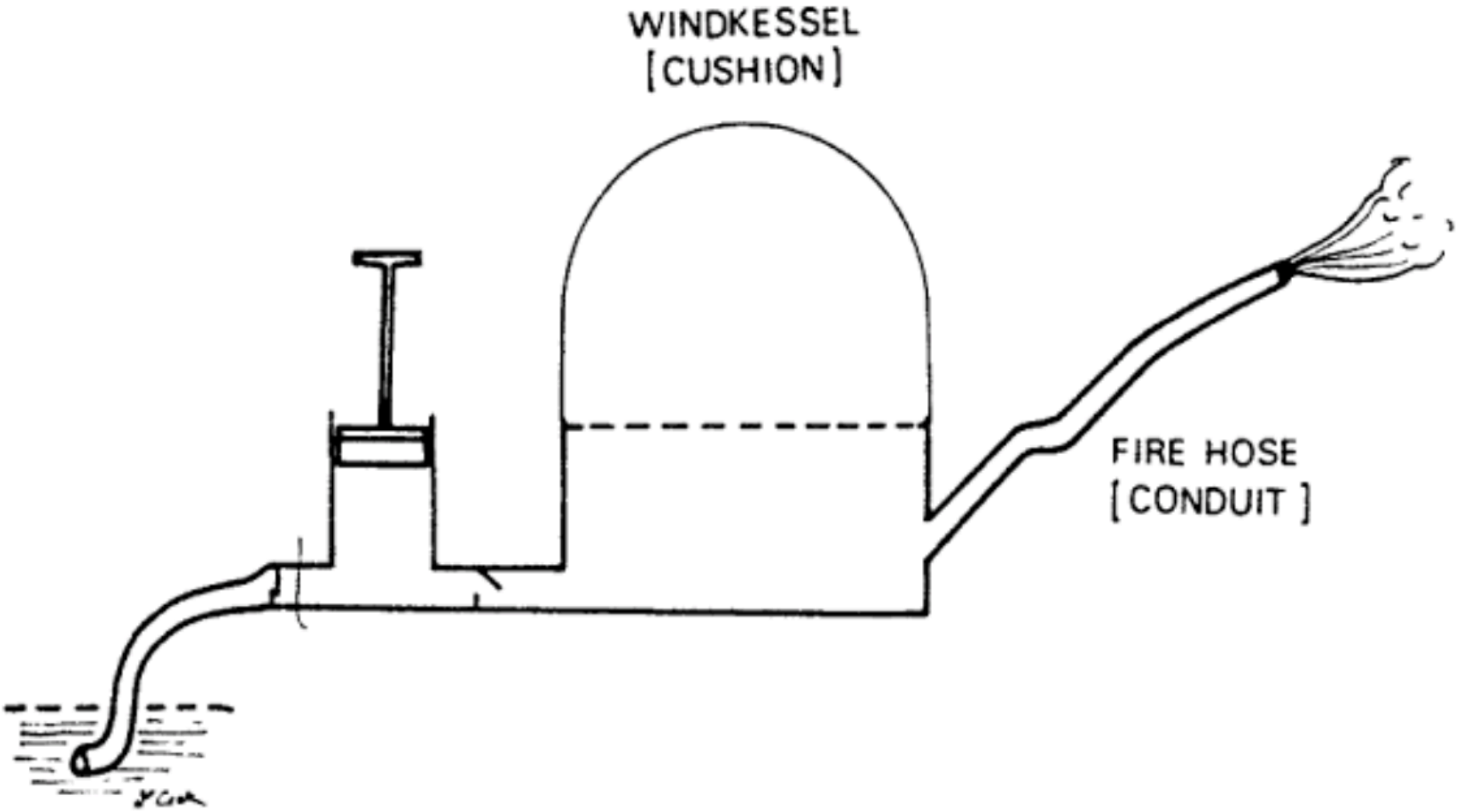
Stergiopulos et al., Am. J. Physiol., 1998



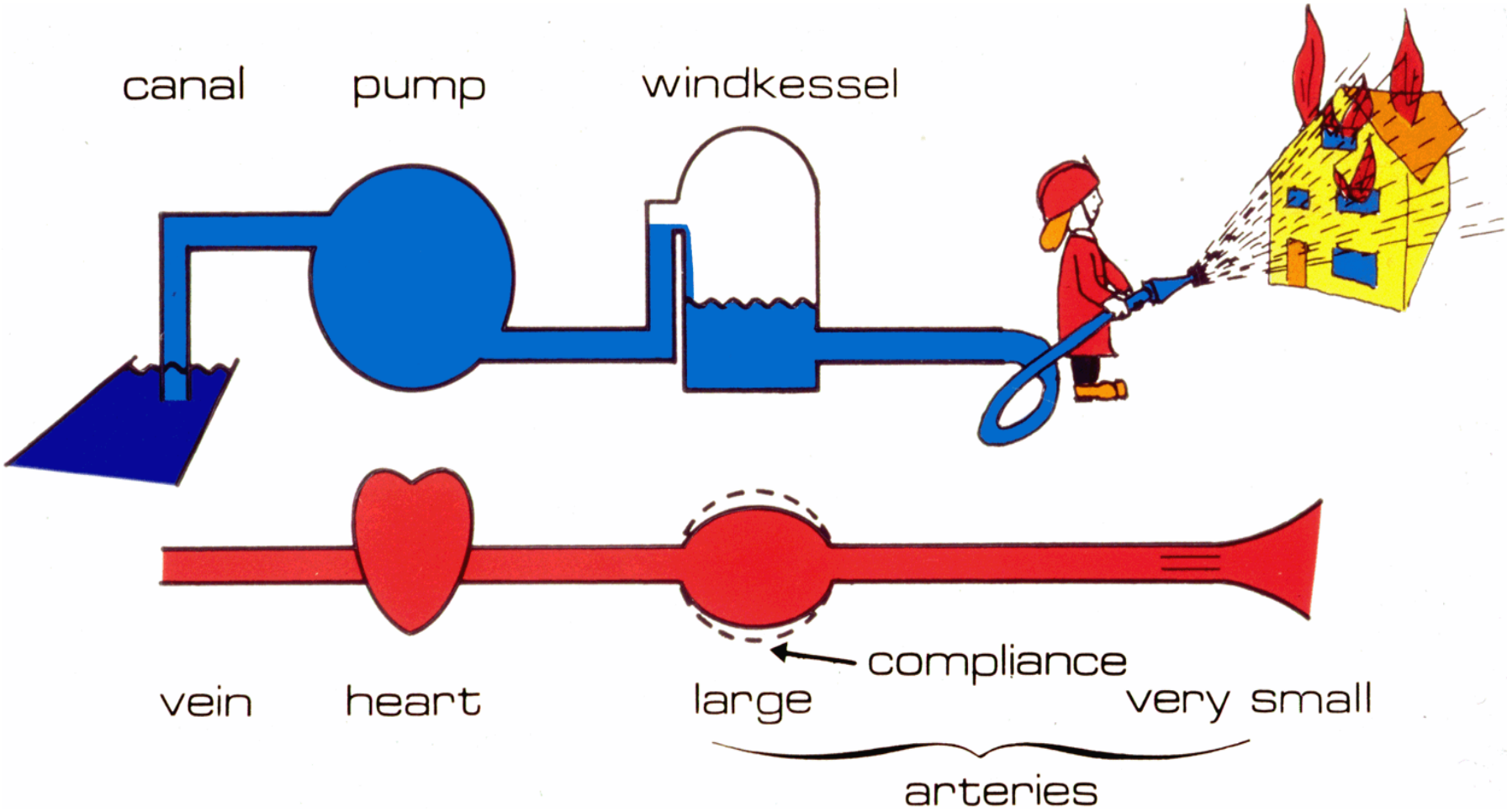
Brachial to aorta



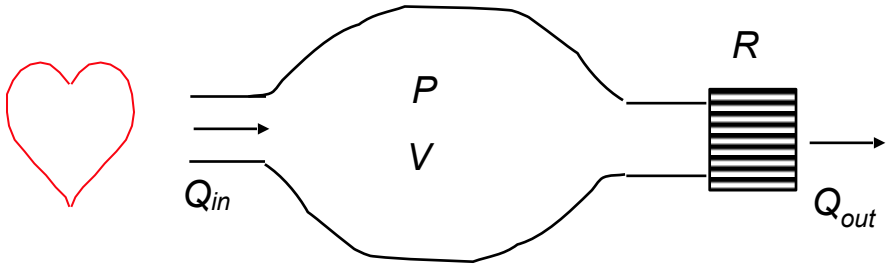
Fire Engine as model



Fire Engine as model



2-element windkessel model (Otto Frank, 1899)

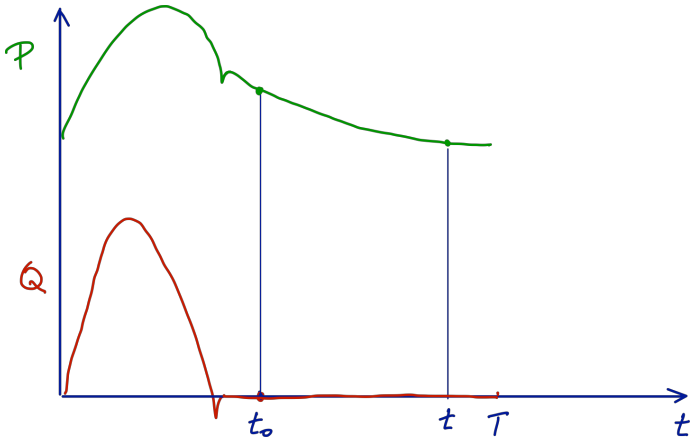


$$R = \frac{P}{Q_{out}}$$

$$C = \frac{dV}{dP}$$

Continuity: $Q_{in} - Q_{out} = \frac{dV}{dt} = \underbrace{\frac{dV}{dP}}_{\text{Volume compliance}} \frac{dP}{dt} = C\frac{dP}{dt} \Rightarrow \underline{\underline{C\frac{dP}{dt} + \frac{P}{R} = Q_{in}}}$ (time domain)

During diastole: $Q_{in} = 0$

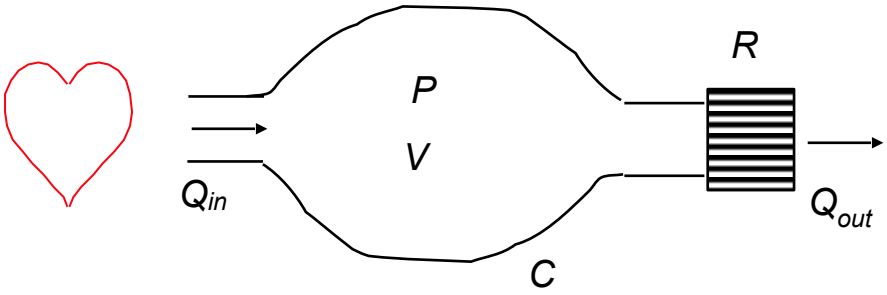


$$\Rightarrow C\frac{dP}{dt} + \frac{P}{R} = 0 \Rightarrow \frac{dP}{dt} = -\frac{P}{RC}$$

$$\Rightarrow \frac{dP}{P} = -\frac{dt}{RC} \Rightarrow \int_{P_0}^{P}\frac{dP}{P} = \int_{t_0}^{t}\frac{dt}{RC}$$

$$\Rightarrow \ln\frac{P}{P_0} = -\frac{t-t_0}{RC} \Rightarrow P(t) = P_0\, e^{-\frac{t-t_0}{RC}}$$

2-element windkessel model (Otto Frank, 1899)



$$R = \frac{P}{Q_{out}}$$

$$C = \frac{dV}{dP}$$

$$C\frac{dP}{dt} + \frac{P}{R} = Q_{in}$$

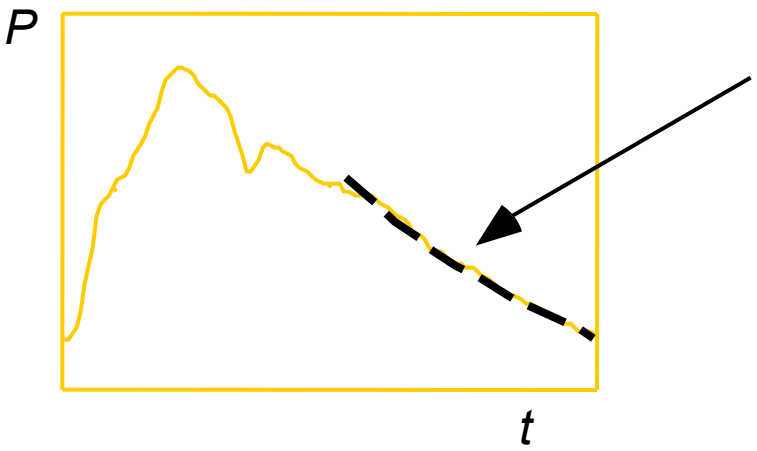
Method to derive total arterial compliance:

a) Fit an exponential decay function to the diastolic part of the curve

b) The characteristic time constant $\tau = RC$

c) Calculate total peripheral resistance as mean pressure over mean flow: $R = \frac{\bar{P}}{\bar{Q}}$

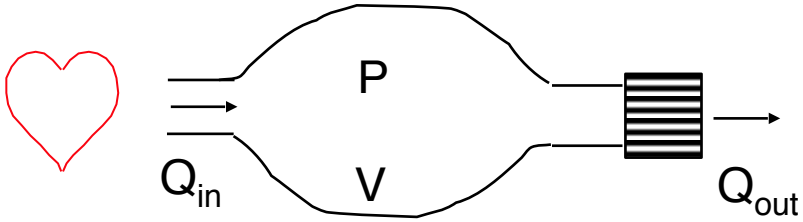
d) Calculate total arterial compliance as: $C = \frac{\tau}{R}$



$$P = P_o e^{-\frac{t}{RC}}$$

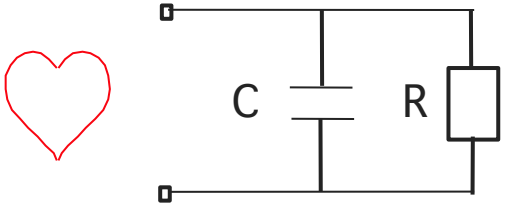
$\tau = RC$ (characteristic time constant)

Electrical analog of the 2-element windkessel model



Time domain: $C\frac{dP}{dt}+\frac{P}{R}=Q$

Electrical analog



Total impedance, Z_{in} is an addition in parallel of R and C

$$\frac{1}{Z_{in}} = \frac{1}{R} + \frac{1}{\frac{1}{j\omega C}} = \frac{1}{R} + j\omega C = \frac{1+j\omega RC}{R}$$

$$\Rightarrow Z_{in} = \frac{R}{1+j\omega RC} \quad \text{(frequency domain)}$$

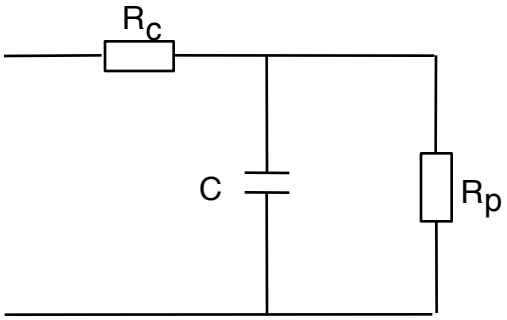
Check if time domain description is the same as the frequency domain one:

$$Z_{in} = \frac{P}{Q} = \frac{R}{1+j\omega RC} \quad \text{(frequency domain)} \Rightarrow P + j\omega RCP = R\cdot Q$$

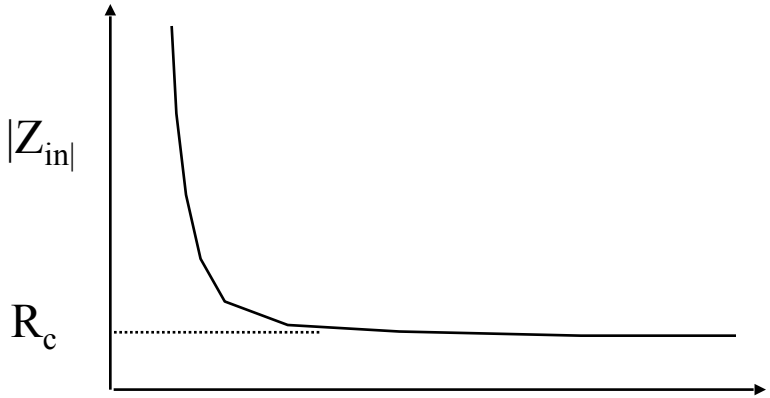
$$\left.\begin{aligned} &P + j\omega RCP = R\cdot Q \\ &\text{Note: } j\omega P = \frac{dP}{dt} \end{aligned}\right\} \Rightarrow C\frac{dP}{dt} + \frac{P}{R} = Q \quad \text{(time domain)}$$

Three-element windkessel

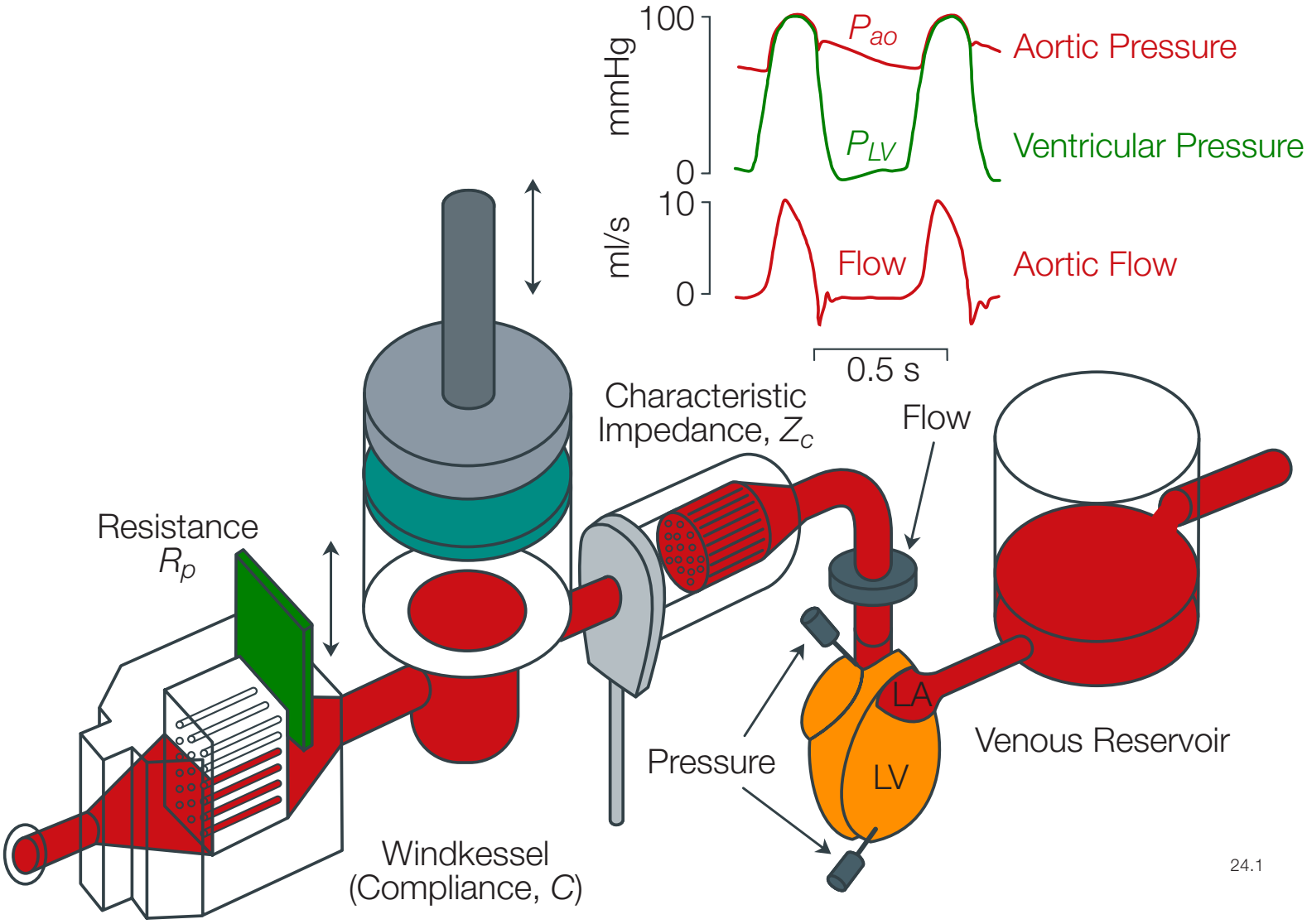
Three-element windkessel



$$Z_{in} = R_c + \frac{R_p}{1 + j\omega R_p C}$$

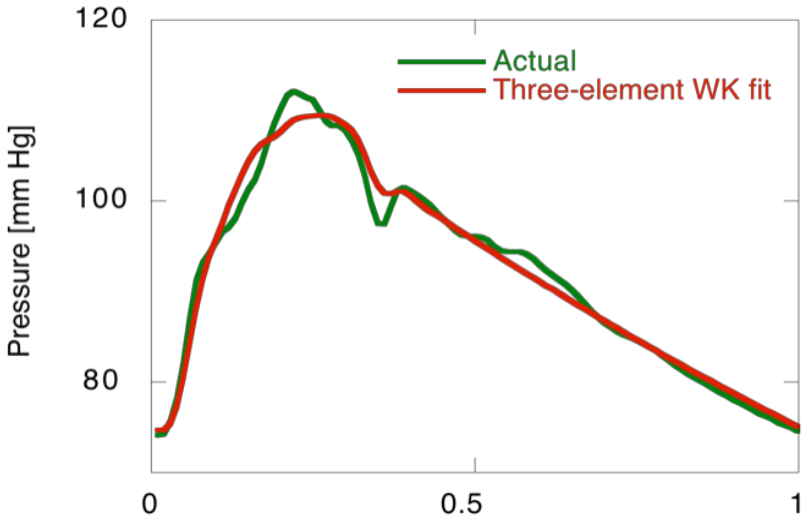


3-element windkessel

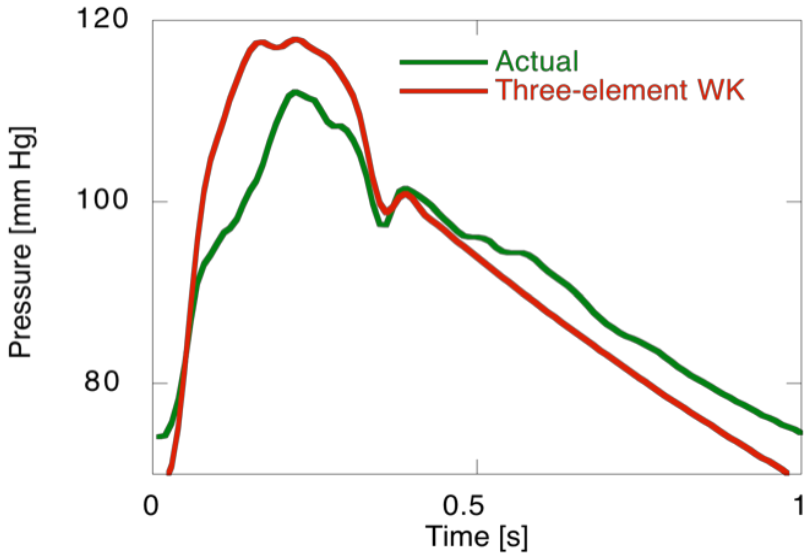


The problem with the 3-element WK

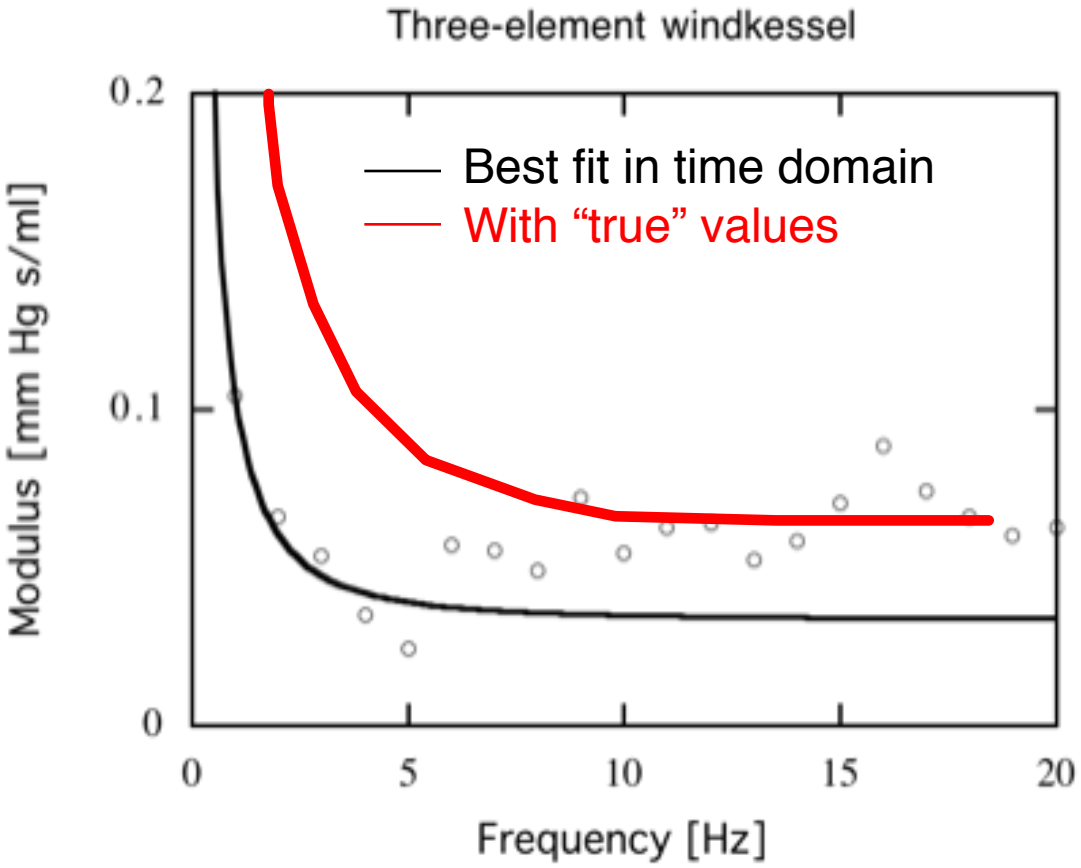
Best fit



With “true” parameters



Why the 3-element model falls short

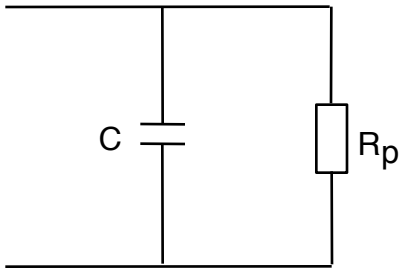


Characteristic impedance forces low-to-mid frequency input impedance to be much higher. Hence in a time domain fit this leads to estimates of:

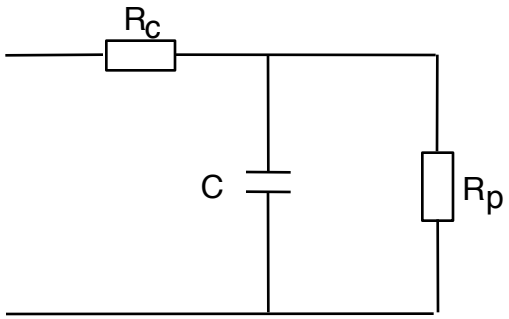
1.Higher compliance
2. Lower characteristic impedance

WIndkessel models

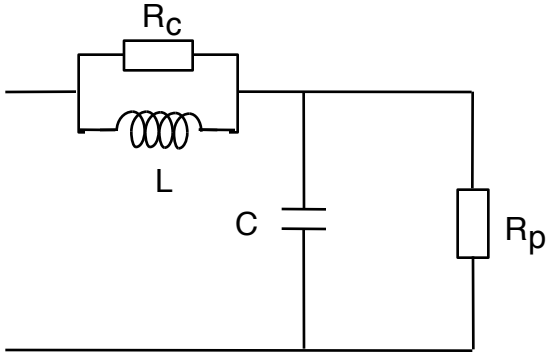
Two-element windkessel



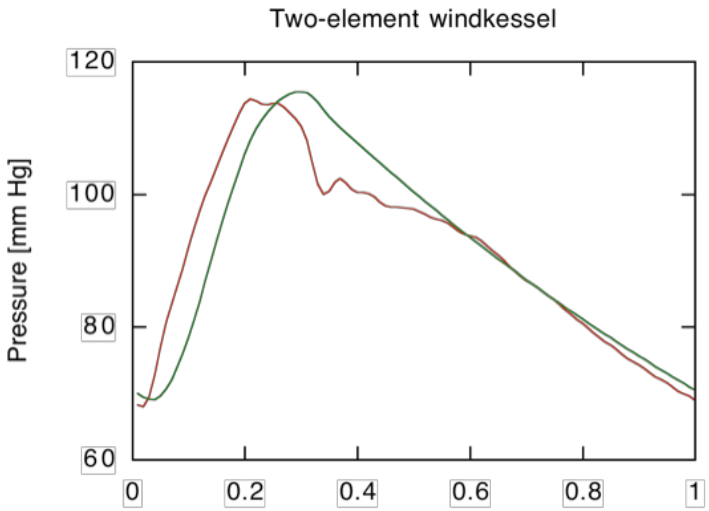
Three-element windkessel

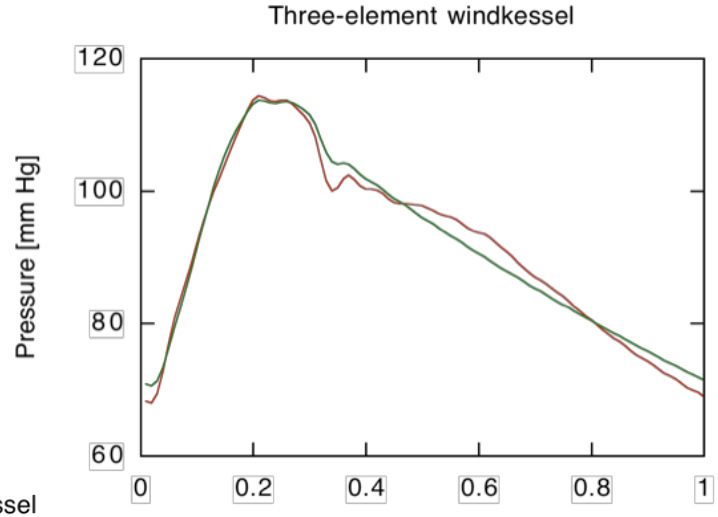


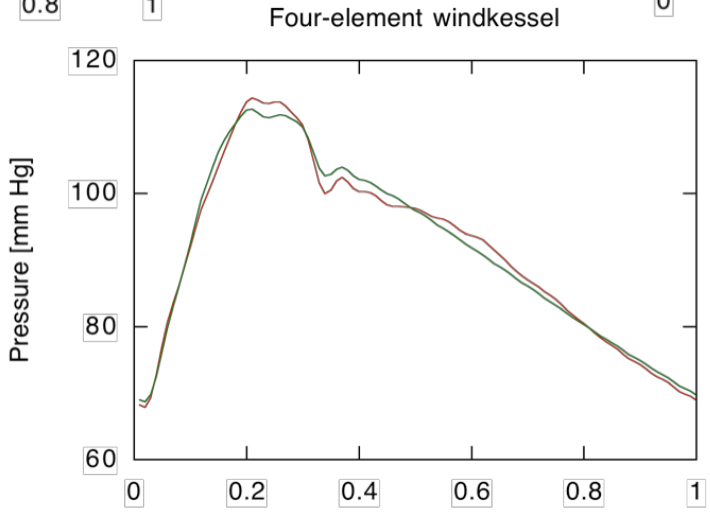
Four-element windkessel



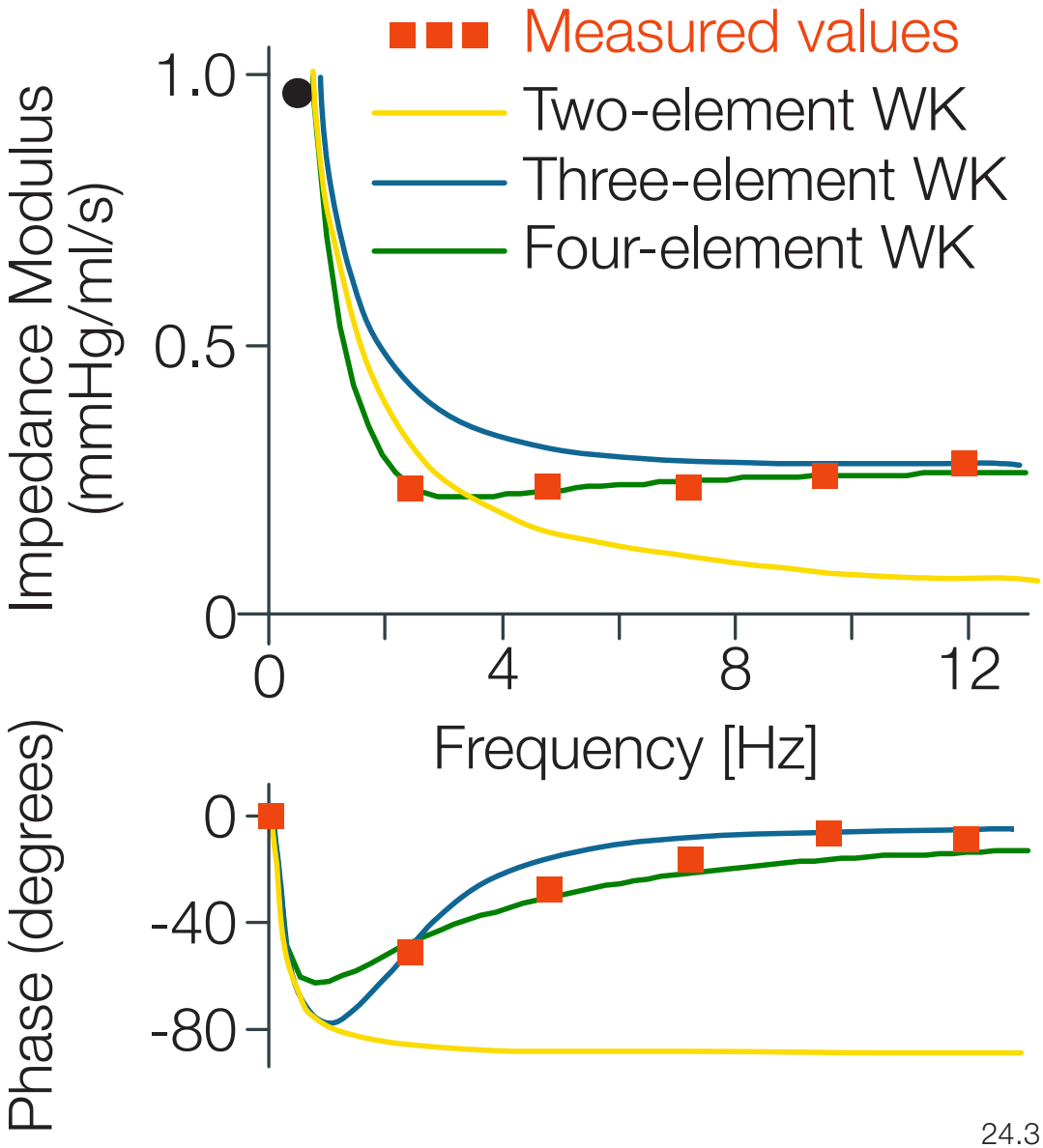
Fit in time domain



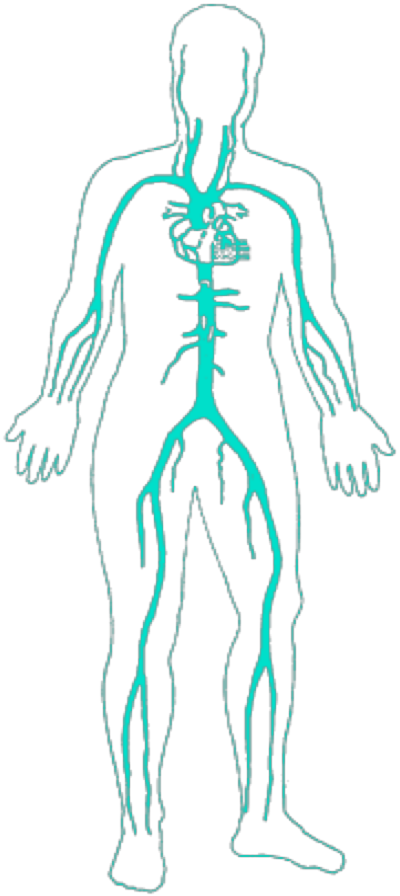




Comparison of input impedances



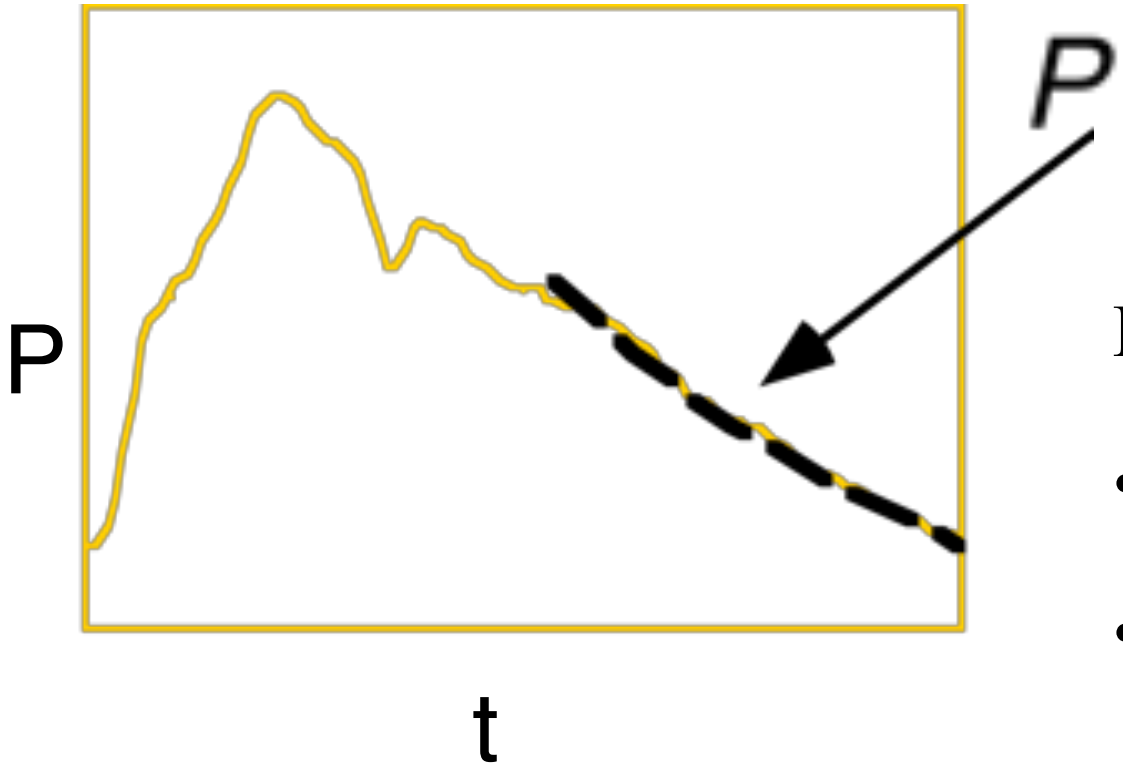
Arterial compliance



$$C = dV/dp$$

- State of the arterial tree
- Heart load and performance
- Wave propagation and reflections
- Relation to arterial disease

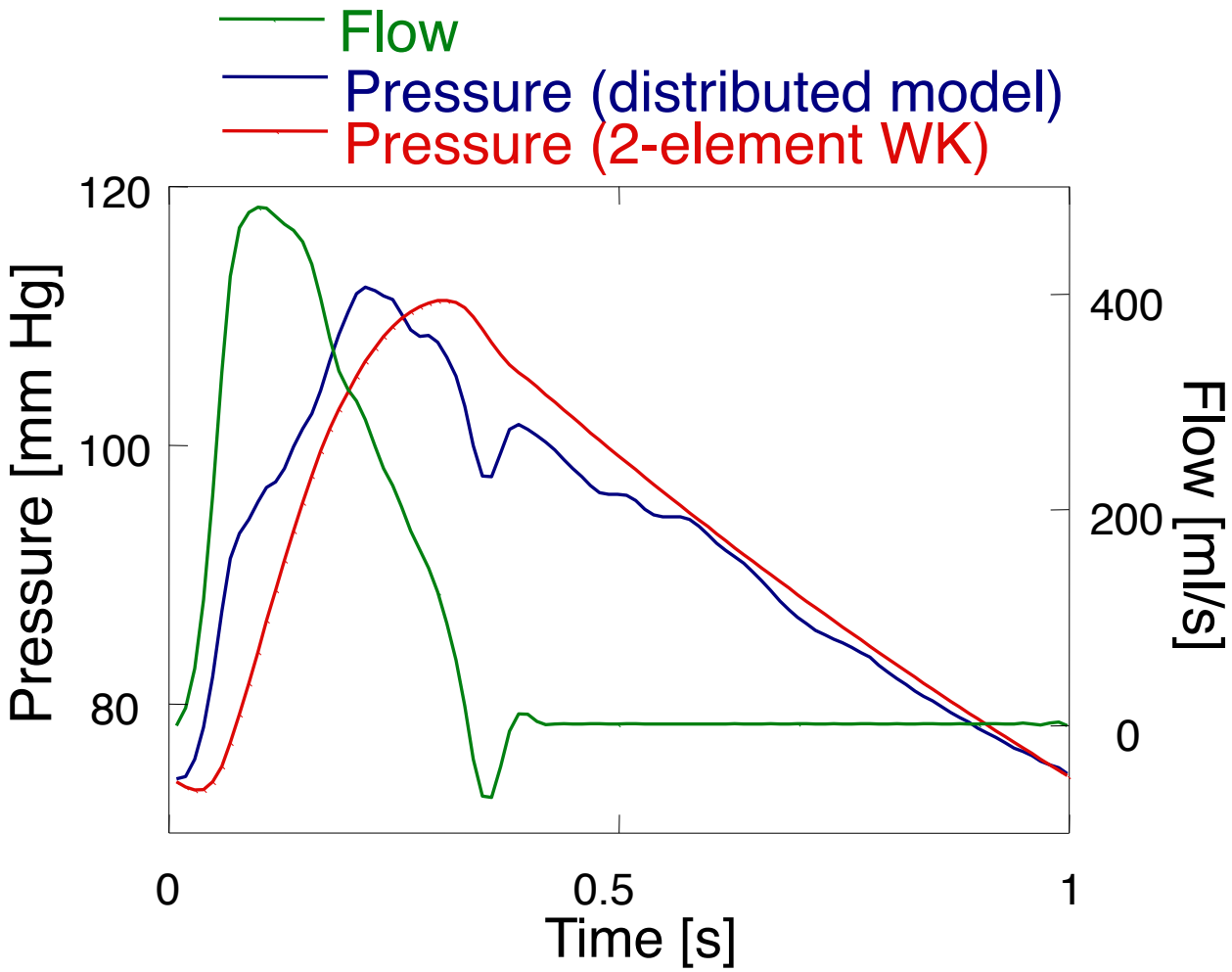
Classic decay time method



Limitation: Q = 0

- Ascending aorta only
- Regurgitation

Distributed vs 2-el Wk model



Distributed vs 2-el WK